DIFFERENTIAL EQUATIONS.

Solved exercises.

1. Find the set of all solutions of the following first order differential equations:

$(a)\ x' = 2t$	$(b)\ y' = 2xy$	$(c)\ x' = 2x$
$(d)\ x' = \frac{1}{2}$	$(e)\ x' = \frac{1}{2t}$	$(f)\ x' = \frac{1}{2t}x$
$(g)\ x' = x^2 \log t$	$(h)\ x' = e^t x \log x$	$(i)\ y' = \frac{x^2-5x+6}{1+t^2}$
$(l)\ y' = 2xy^3$	$(m)\ x' = 2tx + 1$	$(n)\ x' = 2tx + t$
$(o)\ y' = 1 + \frac{y}{t}$	$(p)\ y' = \frac{ty}{t^2+y^2}$	$(q)\ y' = \frac{y(\log y - \log x)}{x}$

2. Find the set of all solutions of the following differential equations of order higher than 1:

$(a)\ y'' = y'$	$(b)\ y'' = \frac{1}{x}y'$	$(c)\ y'' - 2y' - 3y = 0$
$(d)\ x'' = -2x$	$(e)\ x'' = -x - 1$	$(f)\ x'' = -x - 2$

3. Find the solution of the differential equation $x' = -te^{-x}$ tale che $x(0) = -1$. Find its maximal interval of definition. Moreover, compute the maximum value of such function.

4. Given the differential equation

$$x' = \tan^2 t \sqrt[3]{2x+3}$$

1. Find all constant solutions
2. Find for which pairs (t_0, x_0) the hypothesis of the Existence and uniqueness Theorem for the Cauchy problem with initial conditions $x(t_0) = x_0$ are not satisfied. For such pairs are the hypothesis of Peano Existence Theorem satisfied, instead ?
3. Find the set of all solutions.
4. Find the solutions of the Cauchy problem with initial conditions $x(0) = 0$.

5. Given the differential equation

$$x' = \frac{3x-2}{t^2+1}$$

1. Find all constant solutions.
2. Establish whether there exist pairs (t_0, x_0) for which the assumptions of the Existence and uniqueness Theorem of solutions of the Cauchy problem with initial conditions $x(t_0) = x_0$ are not satisfied.
3. Find the set of all solutions.
4. Find the solution of the Cauchy problem with initial conditions $x(0) = 1$, and compute its maximal interval of definition.
5. Find the solution which tends to 0 as $t \to +\infty$.

6. Given the differential equation

$$y' = y^2 - 9y + 8$$

1. Find, if they exist, its constant solutions.
2. Say, without computations, whether there are bounded solutions.
3. Find the solution of the Cauchy problem with initial conditions $y(0) = y_0$.
4. Find the set of all initial values y_0 such that the corresponding solution is defined for all $x \in \mathbb{R}$.

7. Given the differential equation

$$y' = t^3(y^2 + 1)(3 + 4\arctan y)^2$$

1. say whether there esist constant solutions.
2. find the solution of the Cauchy problem with initial conditions $y(0) = 0$, and compute its maximal interval of definition.

8. Find the solutions of the Cauchy problem

$$\begin{cases} t^2 x' = t^2 + 4x^2 + tx \\ x(1) = 0 \end{cases}$$

and determine their domains.

9. Given the differential equation

$$y' = e^{x+y}.$$

1. Find the set of all solutions.
2. compute the maximal interval of definition of the solutions of the Cauchy problem $y(x_0) = y_0$, for all $(x_0, y_0) \in \mathbb{R}^2$.

10. Given the differential equation

$$y' = 2x\sqrt{1 - y^2}.$$

1. For which initial conditions $y(x_0) = y_0$ the Existence Theorem is valid?
2. For which initial conditions $y(x_0) = y_0$ the Existence and Uniqueness Theorem is valid?
3. Find the set of all solutions of the given equation.
4. Find the solutions of the Cauchy problem with initial conditions $y(0) = 1$.

11. Find the functions of class $C^1(\mathbb{R})$ such that:

$$\begin{array}{ll} y' = x\log(1 + x^2) & x \le 1 \\ y' = \frac{1}{x}y - \frac{3x+2}{x^2} & x \ge 1 \end{array}$$

12. Given the differential equation

$$y' = \frac{2t - y}{t - 1}$$

1. Which intervals may be the domain of definition of a solution? For the given equation is it possible to expect, a priori, everywhere defined solutions ?

2. Find the set of all solutions.

3. Find the solutions of the Cauchy problem with initial condition $y(0) = 1$.

4. Find the solutions of the Cauchy problem with initial conditions $y(2) = 1$.

13. Given the differential equation

$$y'' + 3y' - 10y = f(t),$$

compute the general integral of the equation when

(a)	$f(t) = 0$	(b)	$f(t) = 5$
(c)	$f(t) = 3 + 2t$	(d)	$f(t) = 3e^t$
(e)	$f(t) = 2e^{-5t}$	(f)	$f(t) = 2e^{-5t} + 3e^t$.

14. Given the differential equation

$$y'' + 9y = f(t),$$

compute the general integral of the equation when

(a)	$f(t) = 0$	(b)	$f(t) = 4t$
(c)	$f(t) = e^{3t}$	(d)	$f(t) = 4\cos 3t$
(e)	$f(t) = 4\cos 3te^{3t}$	(f)	$f(t) = 4\cos 3t + e^{3t}$

15. Given the differential equation

$$y'' + 4y' + 5y = f(t)$$

with

$$(a)\ f(t) = 0 \qquad (b)\ f(t) = \sin t \qquad (c)\ f(t) = e^{-2t}\sin t$$

1. compute the general integral of the equation;

2. say whether there exist solutions which are bounded over $\mathbb{R}$, and if yes, express them explicitly;

3. say whether there exist solutions which are bounded over $(0, +\infty)$, and if yes, express them explicitly;

4. say whether there exist solutions which tend to 0 as $t \to +\infty$, and if yes, express them explicitly.

16. Let $y_1(t)$ and $y_2(t)$ be solutions of the equation $x'' + bx' + cx = f(t)$ which are defined in an interval I and let $y_3(t)$ be a solution over I of the equation $x'' + bx' + cx = g(t)$.

Prove that:

1. $y_1(t) + y_2(t)$ is a solution of $x'' + bx' + cx = 2f(t)$.

2. $y_1(t) - y_2(t)$ is a solution of the associated homogeneous equation.

3. $y_1(t) + y_3(t)$ is a solution of $x'' + bx' + cx = f(t) + g(t)$.

17. Given the differential equation

$$x'' + 4x' - 5x = b(t),$$

Say whether the following propositions are true or false, and explain why:

1. $x(t) = c_1 e^{-t} + c_2 e^{5t}$, $c_1, c_2 \in \mathbb{R}$ is the general integral of the associated homogeneous equation.
2. If $b(t) = e^{-5t}$, then $y(t) = Ke^{-5t}$ is a solution of the given equation, for some value of $K \in \mathbb{R}$.
3. If $b(t) = 2t + 1$, there exist $A, B \in \mathbb{R}$ such that $y(t) = At + B$ is a solution of the given equation.
4. If $b(t) = 3$, $y(t) = -\frac{3}{5}$ is a solution of the given equation.

SOLUTIONS

Exercise 1. 1a. The solutions are the primitive functions of $2t$, that is $x(t) = t^2 + C$, with $C \in \mathbb{R}$.

1b. The equation is linear and homogeneous of the type $\frac{dy}{dx} = a(x)y$, with $a(x) = 2x$ continuous in $\mathbb{R}$. The solutions are all the functions $y(x) = Ke^{A(x)}$, where $A(x) = x^2$ is a primitive function of $a(x) = 2x$. Hence, the set of all solutions of the given equation is:

$$y(x) = Ke^{x^2}, \qquad K \in \mathbb{R}.$$

1c. The equation is linear and homogeneous; its solutions are $x(t) = Ce^{2t}$, with $C \in \mathbb{R}$.

1d. The solutions are the primitive functions of $\frac{1}{2}$, that is the functions $x(t) = \frac{1}{2}t + C$, as $C \in \mathbb{R}$.

1e. The solutions are the primitive functions of $\frac{1}{2t}$, defined in the interval $(-\infty, 0)$ or in the interval $(0, +\infty)$:

$$x(t) = \tfrac{1}{2}\log t + C, \qquad t > 0, \quad C \in \mathbb{R}$$

$$x(t) = \tfrac{1}{2}\log(-t) + K, \qquad t < 0, \quad K \in \mathbb{R}.$$

1f. The equation is linear and homogeneous $x' = \frac{dx}{dt} = \frac{1}{2t}x$. Since $\frac{1}{2t}$ is defined and continuous in the half-lines $(-\infty, 0)$ and $(0, +\infty)$, we expect to find two families of solutions, each defined in one half-line. To compute the solutions, we may use the formula $x(t) = Ce^{A(t)} = Ce^{\frac{1}{2}\log|t|} = C\sqrt{|t|}$, with $C \in \mathbb{R}$.

In this way, we find the two families of solutions:

$$\begin{array}{lll} x(t) = C\sqrt{t}, & t \in (0, +\infty) & C \in \mathbb{R} \\ x(t) = K\sqrt{-t}, & t \in (-\infty, 0) & K \in \mathbb{R} \end{array}$$

1g. The equation has separable variables, since it is of the form $\frac{dx}{dt} = a(t)b(x)$, with $a(t) = \log t$ defined and continuous for all $t > 0$, $b(x) = x^2$ defined and of class C^1 for all $x \in \mathbb{R}$.

Constant solutions: $b(x) = x^2 = 0$ is and only if $x = 0$. Hence, the function $x(t) = 0$ is a solution of the equation for all $t \in (0, +\infty)$.

Other solutions. Upon integration we get

$$\int \frac{1}{x^2}dx = \int \log t dt \qquad \text{if and only if} \quad -\frac{1}{x} = t\log t - t + C.$$

Expressing x as a function of t we get

$$x(t) = \frac{1}{t - t\log t + K}, \qquad K \in \mathbb{R}.$$

These functions, each considered in the interval contained in $(0, +\infty)$ where they are defined, are the solutions of the equation, together with the constant solution $x(t) = 0$, already determined.

1h. The equation has separable variables, $a(t) = e^t$ is continuous in $\mathbb{R}$, $b(x) = x\log x$ is defined and of class C^1 for all $x > 0$. *Constant solutions:* $b(x) = x\log x = 0$ if and only if $x = 1$. Hence, the function $x(t) = 1$ is a solution of the equation for all $t \in \mathbb{R}$.

Other solutions. After integration we get

$$\int \frac{1}{x\log x}dx = \int e^t dt \qquad \text{if and only if} \quad \log|\log x| = e^t + C, \quad C \in \mathbb{R}$$

It follows that

$$|\log x| = e^{e^t+C} = e^C e^{e^t} = Ke^{e^t}, \qquad K > 0,$$

that is $\log x = \pm Ke^{e^t}$, with $K > 0$. Noting that for $K = 0$ we obtain the constant solution, we have that the set of all solutions is:

$$x(t) = e^{Ce^{e^t}}, \qquad \forall t \in \mathbb{R}, \quad \forall C \in \mathbb{R}.$$

1i. The equation has separable variables, $a(t) = \frac{1}{1+t^2}$ is continuous for $t \in \mathbb{R}$, $b(x) = x^2-5x+6$ of class C^1 in $\mathbb{R}$.

Constant solutions: $b(x) = x^2 - 5x + 6 = (x-3)(x-2) = 0$ is and only if $x = 2$ or $x = 3$. Hence, the functions $x(t) = 2$ and $x(t) = 3$ are solutions of the equation for every $t \in \mathbb{R}$.

Since the assumptions of the Existence and Uniqueness Theorem are satisfied for every pair $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}$, we note that every solution satisfying the initial condition $x(t_0) = x_0$, for $x_0 \in (2, 3)$, is bounded since its graph cannot intersect the straight lines $x(t) = 2$ e $x(t) = 3$, corresponding to solutions of the same equation.

The other solutions. After integration we get

$$\log\left|\frac{x-3}{x-2}\right| = \arctan t + C, \quad C \in \mathbb{R}$$

and so

$$\left|\frac{x-3}{x-2}\right| = Ke^{\arctan t}, \quad K > 0$$

Setting $K \neq 0$ we can eliminate the modulus. Moreover, if $K = 0$ we obtain again the constant solution $x(t) = 3$. In order to express the solution explicitly we note that $\frac{x-3}{x-2} = \frac{x-2-1}{x-2} = 1 - \frac{1}{x-2}$. Hence $\frac{1}{x-2} = 1 - Ke^{\arctan t}$, $K \in \mathbb{R}$. Hence, the solutions of the given equation are

$$x(t) = 2 + \tfrac{1}{1-Ke^{\arctan t}}, \qquad K \in \mathbb{R}$$

$$x(t) = 2.$$

1l. The equation has separable variables, i.e., it is of the type $\frac{dy}{dx} = a(x)b(y)$, with $a(x) = 2x$ continuous in $\mathbb{R}$ and $b(y) = y^3$ of class C^1 in $\mathbb{R}$.

Constant solutions: $b(y) = y^2 = 0$ if and only if $y = 0$. Hence, the only constant solution is the function $y(x) = 0$, with $x \in \mathbb{R}$.

The other solutions. After integration we get

$$-\frac{1}{2}\cdot\frac{1}{y^2} = x^2 + C, \qquad C \in \mathbb{R}.$$

In order to express y as a function of x, we note that $\frac{1}{y^2} = -2x^2 + K$, with $K \in \mathbb{R}$, from which we have $y^2 = 1/(-2x^2 + K)$. Taking the square root we find the set of solutions which is:

$$y(x) = 0$$
$$y(x) = \pm\tfrac{1}{\sqrt{-2x^2+K}}, \qquad K \in \mathbb{R},$$

where the last two functions are defined only in the intervals where $-2x^2 + K > 0$.

1m. The equation is linear and non homogeneous; its solutions can be obtained by the formula $x(t) = e^{t^2} \int e^{-t^2} dt$. Since the primitive function of e^{-t^2} cannot be expressed in terms of elementary functions, then also the solutions to such equation cannot be expressed by means of elementary functions.

1n. The equation is linear and non homogeneous; but in this case $x(t) = e^{t^2} \int te^{-t^2} dt = e^{t^2}(\frac{1}{2}e^{-t^2} + C) = \frac{1}{2} + Ce^{t^2}$, with $C \in \mathbb{R}$.

1o. The equation is homogeneous; in fact $1 + \frac{y}{t}$ depends on $\frac{y}{t}$. Setting $z = \frac{y}{t}$, and noting that $tz' + z = y'$ the equation takes the form $z' = \frac{1}{t}$, whose solutions are $z(t) = \log|t| + C$, with $C \in \mathbb{R}$. Hence, the solutions of the given equation are:

$$y(t) = t \log|t| + Ct, \qquad C \in \mathbb{R},$$

which are defined either in the interval $(-\infty, 0)$ or in the interval $(0, +\infty)$.

1p. The equation is homogeneous; in fact the right-hand side $f(t, y)$ is the quotient of two polynomials homogeneous of the same degree. Setting $z = \frac{y}{t}$, and noting that $tz' + z = y'$, the equation takes the form of an equation with separable variables:

$$z' = \frac{1}{t} \cdot \left[\frac{z}{1 + z^2} - z\right] = -\frac{z^3}{1 + z^2} \cdot \frac{1}{t}.$$

Constant solutions. Let us note that the function $y(t) = 0$ is a solution, which is defined for every $t \in \mathbb{R}$.

The other solutions. Integrating the equation with respect to z, we get

$$-\frac{1}{2} \cdot \frac{1}{z^2} + \log|z| = -\log|t| + C, \qquad C \in \mathbb{R}.$$

This equation cannot be trivially solved with respect to z: hence, we simply replace the original variables, leaving the solutions in an implicit form:

$$-\frac{1}{2} \cdot \frac{t^2}{y^2} + \log\left|\frac{y}{t}\right| = -\log|t| + C, \qquad C \in \mathbb{R}.$$

These functions together with the constant solution $y(t) = 0$ form the set of all the solutions of the given equation.

1q. First of all, let us note that the equation is defined only in the set $\{(x, y) \in \mathbb{R}^2 : x > 0 \textit{ and } y > 0\}$. In this set, the equation can be written also as $y' = \frac{y}{x} \log \frac{y}{x}$. Replacing $z = \frac{y}{x}$ and working as in the preceding two exercises, we get a new equation with separable variables:

$$z' = \frac{1}{x}(z \log z - z) = \frac{1}{x} z(\log z - 1).$$

Let us note that in this case we have to impose the following conditions: $z > 0$ e $x > 0$.

Costant solutions. The right-hand side of the equation is zero if we replace $z = 0$ or $\log z = 1$, but only the second choice is compatible with the condition $z > 0$. Hence we have that $z = \frac{y}{x} = e$ is a solution of the equation. And so:

$$y(x) = ex, \qquad x \in (0, +\infty)$$

is a solution of the given equation.

The other solutions. Integrating the equation with respect to z, we get

$$\log|\log z - 1| = \log|x| + C = \log|x| + \log K = \log K|x|, \qquad K > 0,$$

from which we obtain

$$\log z - 1 = \log\frac{y}{x} - 1 = Kx, \qquad \text{con } K > 0 \text{ e } x > 0.$$

The set of all solutions is then:

$$y(x) = xe^{Cx+1}, \qquad \text{con } C \in \mathbb{R} \text{ e } x > 0.$$

Exercise 2.

2a. Let us set $z = y'$. In this way, the equation transforms into a linear equation of the first order $z' = z$, whose solutions are $z(t) = Ce^t$, for all $C \in \mathbb{R}$. In order to find the solutions of the given equation, we can integrate:

$$y(t) = \int Ce^t dt = Ce^t + K, \qquad C, K \in \mathbb{R}.$$

The equation can be solved also as a second order linear equation with constant coefficients. The solutions are obviously the same.

2b. Let us note that $y'' = f(x, y, y') = \frac{1}{x}y'$, where f is not defined at $x = 0$. Hence, also the expected solutions should be defined in intervals excluding $x = 0$.

Let us set $z = y'$. In this way, the equation takes the form of a linear equation of the first order $z' = \frac{1}{x}z$, which has solutions $z(t) = Ce^{\log|x|} = C|x|$, for all $C > 0$. Setting $C \in \mathbb{R}$ we can eliminate the modulus, remembering that the solutions can be defined only in intervals excluding $x = 0$. To get the solutions of the given equation, we can integrate:

$$y(x) = \int Cx dx = \frac{1}{2}Cx^2 + K, \qquad C, K \in \mathbb{R}.$$

2c. The equation is linear of the second order, with constant coefficients. Its characteristic equation is $z^2 - 2z + 3 = 0$, which has $\Delta < 0$ and two complex solutions $z = 1 \pm \sqrt{-2}$. We get two real and linearly independent solutions of the differential equation $y_1(t) = e^t \cos\sqrt{2}t$, $y_2(t) = e^t \sin\sqrt{2}t$. The general integral is:

$$y(t) = c_1 e^t \cos\sqrt{2}t + c_2 e^t \sin\sqrt{2}t, \qquad c_1, c_2 \in \mathbb{R}.$$

2d. Writing the equation as $x'' + 2x = 0$, we recognize the harmonic oscillator, with $\omega = \sqrt{2}$. The general integral is

$$x(t) = c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t, \qquad c_1, c_2 \in \mathbb{R}.$$

2e. Writing the equation as $x'' + x = -1$, we recognize the harmonic oscillator, with $\omega = 1$ and forcing term $f(t) = -1$. It is easy to see that $x(t) = -1$ is a solution of the non homogeneous equation. Hence, the general integral is

$$x(t) = c_1 \cos t + c_2 \sin t - 1, \qquad c_1, c_2 \in \mathbb{R}.$$

2f. This equation is of the same tipe as in the preceding exercise: it has the constant solution $x(t) = -2$.

Exercise 3. The equation has separable variables, with con $a(t) = -t$, $b(x) = e^{-x} \neq 0$. Both functions are continuous and differentiable in $\mathbb{R}$, hence, for all pairs (t_0, x_0), and especially for $t_0 = 0$ and $x_0 = -1$ the assumptions of the Existence and uniqueness Theorem are satisfied. After integration we get $e^x = -\frac{1}{2}t^2 + C$: the solutions are $x(t) = \log\left(-\frac{1}{2}t^2 + C\right)$. By imposing the initial condition, we get $x(0) = \log C = -1$. Hence $C = e^{-1}$, and the solution to the Cauchy problem is:

$$x(t) = \log\left(-\frac{1}{2}t^2 + e^{-1}\right).$$

This function is defined for $-\frac{1}{2}t^2 + e^{-1} > 0$. The maximal interval of definition is $(-\sqrt{2e^{-1}}, \sqrt{2e^{-1}})$.

Deriving $x(t)$, we see that at $t = 0$ the solution attains its maximum value.

Exercise 4. The equation has separable variables, with $a(t) = \tan^2 t$ and $b(x) = \sqrt[3]{2x+3}$.

1. The constant solutions correspond to the zeroes of $b(x)$. Hence, there is only one constant solution as $x = -\frac{3}{2}$.

2. $a(t)$ is defined and continuous in every interval of the type $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$, with $k \in \mathbb{Z}$. The function $b(x)$ is continuous in $\mathbb{R}$, but non differentiable at $x = -\frac{3}{2}$. Hence, the assumptions of the Existence and uniqueness Theorem are satisfied at the points (t_0, x_0), with $t_0 \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$ for some $k \in \mathbb{Z}$, and $x_0 \neq -\frac{3}{2}$.

   On the contrary, the assumptions of the Existence Theorem of Peano are satisfied also at $x_0 = -\frac{3}{2}$, since $b(x)$ is continuous at that point.

3. After integration we get $(2x+3)^{\frac{2}{3}} = \frac{4}{3}(\tan t - t + C)$, with $C \in \mathbb{R}$. Taking the cube we obtain $(2x+3)^2 = \left[\frac{4}{3}(\tan t - t + C)\right]^3$, from which we get

   $$x(t) = -\frac{3}{2} \pm \frac{1}{2}\sqrt{\left[\frac{4}{3}(\tan t - t + C)\right]^3}, \qquad C \in \mathbb{R}.$$

   These functions, together with the constant $x(t) = -\frac{3}{2}$ form the set of all solutions.

4. Let us impose that $x(0) = 0$. we get that

   $$0 = -\frac{3}{2} \pm \frac{1}{2}\sqrt{\left[\frac{4}{3}C\right]^3}.$$

   from which $\frac{3}{2} = \pm\frac{1}{2}\sqrt{\left[\frac{4}{3}(C)\right]^3}$ that reduces to $\frac{3}{2} = +\frac{1}{2}\sqrt{\left[\frac{4}{3}C\right]^3}$. We get $C = \frac{3}{4}\sqrt[3]{3}$.

   The solution of the given Cauchy problem is then

   $$x(t) = -\frac{3}{2} + \frac{1}{2}\sqrt{\left[\frac{4}{3}\left(\tan t - t + \frac{3}{4}\sqrt[3]{3}\right)\right]^3}.$$

Exercise 5. The equation has separable variables, with $a(t) = \frac{1}{1+t^2}$ and $b(x) = 3x - 2$.

1. The only constant solution is $x(t) = \frac{2}{3}$.

2. $a(t)$ and $b(x)$ are of class C^1 in $\mathbb{R}$, hence for all pairs (t_0, x_0) the assumptions of the Cauchy Theorem are satisfied.

3. After integration we get $\frac{1}{3}\log|3x-2| = \arctan t + C$. Multiplying by 3 and taking the exponential, we obtain $|3x-2| = e^{3\arctan t + C} = Ke^{3\arctan t}$, with $K > 0$. To eliminate the modulus, we can let K vary in $\mathbb{R}\setminus\{0\}$. In this way, as $K = 0$ we obtain the constant solution, while if $K \in \mathbb{R}$ from $3x - 2 = Ke^{3\arctan t}$ we obtain all the solutions. Hence, the set of all solutions is

$$x(t) = \frac{2}{3} + ke^{3\arctan t}, \qquad k \in \mathbb{R}.$$

4. Imposing the initial condition we obtain $k = \frac{1}{3}$. The corresponding solution is

$$x = \frac{2}{3} + \frac{1}{3}e^{3\arctan t},$$

which is defined in $\mathbb{R}$.

5. Let us note that

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty}\left(\frac{2}{3} + ke^{3\arctan t}\right) = \frac{2}{3} + ke^{\frac{\pi}{2}} = 0$$

if and only if $k = -\frac{2}{3}e^{-\frac{\pi}{2}}$. The solution is then

$$x(t) = \frac{2}{3} - \frac{2}{3}e^{-\frac{\pi}{2}}e^{3\arctan t} = \frac{2}{3}\left(1 - e^{3\arctan t - \frac{\pi}{2}}\right).$$

Exercise 6. The equation has separable variables, with $a(t) = 1$ and $b(y) = y^2 - 9y + 8$.

1. The constant solutions correspond to the zeroes of $b(y)$: $y(t) = 1$ and $y(t) = 8$ are the constant solutions. They are defined for all $t \in \mathbb{R}$.

2. Let us note that $a(t) = 1$ is continuous in $\mathbb{R}$ and $b(y)$ is of class C^1 in $\mathbb{R}$. Hence, in each point of the plane (t, y) the assumptions of the Theorem of existence and uniqueness are satisfied: this means that the graphs of any two solutions cannot intersect. In particular, the constant solutions $y(t) = 1$ and $y(t) = 8$ cannot intersect other solutions. Hence, any solution with initial condition $y(t_0) = y_0 \in (1, 8)$ should have its graph contained in the stripe $1 < y < 8$ for every t, and so it has to be a bounded solution.

3. Upon integration we get:

$$-\frac{1}{7}\log|y-1| + \frac{1}{7}\log|y-8| = t + C, \qquad C \in \mathbb{R},$$

from which it follows

$$\left|\frac{y-8}{y-1}\right| = Ke^{7t}, \qquad K > 0.$$

Removing the absolute values, and noting that as $K = 0$ we obtain again the constant solution $y(t) = 8$, we have that $\frac{y-8}{y-1} = Ke^{7t}$ for all $K \in \mathbb{R}$ gives all the solutions except $y(t) = 1$. By imposing the initial condition $y(0) = y_0$ we get: $K = \frac{y_0-8}{y_0-1}$, if $y_0 \neq 1$.

The set of solutions for all values of y_0 in $\mathbb{R}$ is:

$$\begin{aligned} y(t) &= 1 + \frac{7}{1 - \frac{y_0-8}{y_0-1}e^{7t}}, &\qquad& \text{if } y_0 \neq 1 \\ y(t) &= 1 && \text{if } y_0 = 1. \end{aligned}$$

4. The constant solutions $y(t) = 1$ and $y(t) = 8$ are defined all over $\mathbb{R}$ (this corresponds to choose as initial condition $y_0 = 1$ or $y_0 = 8$). All the other solutions are defined in $\mathbb{R}$ if $1 - \frac{y_0-8}{y_0-1}e^{7t} \neq 0$ for every $t \in \mathbb{R}$. This is equivalent to impose that the equality $e^{7t} = \frac{y_0-1}{y_0-8}$ is never satisfied. This is true when $\frac{y_0-1}{y_0-8} \leq 0$ (since the exponential function is everywhere positive and has range $]0, +\infty[$). The inequality is satisfied for all $y_0 \in (1, 8)$.

   In conclusion: for every $y_0 \in [1, 8]$ the solution satisfying the initial condition $y(t_0) = y_0$ is defined all over $\mathbb{R}$. For initial conditions out of such interval, the corresponding solution has a smaller domain.

Exercise 7. The equation has separable variables, with $a(t) = t^3$ and $b(y) = (y^2 + 1)(3 + 4\arctan y)^2$.

1. The constant solutions correspond to the zeroes of $b(y)$: $y(t) = \tan\left(-\frac{3}{4}\right) = -\tan\left(\frac{3}{4}\right)$.

2. In order to find the solution of the Cauchy problem (which is unique, being all assumptions of the Existence and uniqueness Theorem satisfied) we start by integrating the equation:

$$\int \frac{1}{(y^2+1)(3+4\arctan y)^2} dy = \int t^3 dt$$

   Replacing $u = 3 + 4\arctan y$, we have that $\frac{1}{4}du = \frac{1}{(y^2+1)}dy$. And so we get:

$$-\frac{1}{4} \cdot \frac{1}{3+4\arctan y} = \frac{1}{4}t^4 + C, \qquad C \in \mathbb{R}.$$

   By imposing the initial condition $y(0) = 0$, we find $C = -\frac{1}{12}$. After some rearrangements, we get:

$$\frac{1}{3+4\arctan y} = -t^4 + \frac{1}{3} = \frac{1-3t^4}{3}.$$

   Let us note that the left-hand side is never zero, while the right-hand side is zero at $t = \pm\sqrt[4]{\frac{1}{3}}$. Hence, the maximal interval of definition of our solution is contained in the interval $\left(-\sqrt[4]{\frac{1}{3}}, \sqrt[4]{\frac{1}{3}}\right)$.
   Let us express our solution explicitly:

$$\arctan y = \frac{1}{4}\left(\frac{3}{1-3t^4} - 3\right).$$

   Since the range of arctan is $]-\pi/2, \pi/2[$, we have to impose that

$$-\frac{\pi}{2} < \frac{1}{4}\left(\frac{3}{1-3t^4} - 3\right) < \frac{\pi}{2}.$$

   The left-hand side inequality is automatically satisfied in the interval $\left(-\sqrt[4]{\frac{1}{3}}, \sqrt[4]{\frac{1}{3}}\right)$, since $1 - 3t^4 < 0$. As for the right-hand side inequality, after some rearrangements we get

$$t^4 < \frac{2\pi}{3(2\pi+3)} < \frac{1}{3}.$$

   Hence, the maximal interval of definition of the solution is $\left(-\sqrt[4]{\frac{2\pi}{3(2\pi+3)}}, \sqrt[4]{\frac{2\pi}{3(2\pi+3)}}\right)$, and the solution is:

$$y(t) = \tan\frac{1}{4}\left(\frac{3}{1-3t^4} - 3\right).$$

Exercise 8. In order to integrate, we should divide by t^2. Let us note that if there exist solutions which are defined at $t = 0$, then they should be 0 at $t = 0$, since, by replacing $t = 0$ in the equation we find necessarily $x = 0$.

Dividing by t^2, the equation transforms into

$$x' = 1 + 4\frac{x^2}{t^2} + \frac{x}{t},$$

which is a homogeneous equation. Let us set now $z = \frac{x}{t}$. It follows $x' = z + tz'$. Replacing:

$$z + tz' = 1 + 4z^2 + z$$

simplifying and dividing by t we get the separable variables equation in z:

$$z' = \frac{1}{t}(1 + 4z^2).$$

This equation does not have constant solutions. Upon integration we get

$$\frac{1}{2}\arctan z = \log|t| + C, \qquad C \in \mathbb{R}.$$

In order to determine C, let us remind that $z = \frac{x}{t}$. The initial condition is $x(1) = 0$. Hence, inserting these values in the equation we get

$$\frac{1}{2}\arctan 0 = \log 1 + C,$$

so that $C = 0$. Hence, the solution of the Cauchy problem in implicit form is

$$\arctan z = 2\log t = \log t^2, \qquad \text{with } \ t > 0.$$

Since the range if arctan is $]-\pi/2, \pi/2[$, we have to impose that $-\frac{\pi}{2} < \log t^2 < \frac{\pi}{2}$. The positive values for which this inequality is satisfied are $\left(e^{-\pi/4}, e^{\pi/4}\right)$.

Expressing the solution in explicit form we have $z = \frac{x}{t} = \tan\log t^2$. Hence, the solution of the given Cauchy problem is:

$$x(t) = t\tan\log t^2, \qquad t \in \left(e^{-\frac{\pi}{4}}, e^{\frac{\pi}{4}}\right).$$

Exercise 9.

1. The equation has separable variables, in fact $y' = e^{x+y} = e^x \cdot e^y$. Upon integration we get $e^{-y} = -e^x - C$, with $C \in \mathbb{R}$. In explicit form, we obtain the set of solutions

   $$y(x) = -\log(-e^x - C), \qquad C \in \mathbb{R}.$$

2. By imposing the initial conditions, we have that $e^{-y_0} = -e^{x_0} - C$, from which it follows that $C = -e^{x_0} - e^{-y_0}$. Hence, the solution of the Cauchy problem is $y(x) = -\log(-e^x + e^{x_0} + e^{-y_0})$, which is defined under the condition $-e^x + e^{x_0} + e^{-y_0} > 0$, that is for all $x < \log(e^{x_0} + e^{-y_0})$.

Exercise 10. The equation has separable variables, being of the type $\dfrac{dy}{dx} = a(x)b(y)$. The function $a(x) = 2x$ is continuous in $\mathbb{R}$, while $b(y)$ is defined and continuous in $[-1, 1]$, and has continuous first derivative in $(-1, 1)$.

1. It follows that the Existence Theorem is satisfied for every $x_0 \in \mathbb{R}$ and every $y_0 \in [-1, 1]$.

2. The Uniqueness Theorem holds true, instead, if $y_0 \in (-1, 1)$.

3. First of all, there are two constant solutions $y(x) = 1$ and $y(x) = -1$, both defined for all $x \in \mathbb{R}$.

   To compute the other solutions we integrate:

$$\int \frac{1}{\sqrt{1-y^1}}\,dy = \int 2x\,dx \iff \arcsin y = x^2 + K, K \in \mathbb{R}$$

   Noting that arcsin is the inverse function of sin restricted to the interval in $[-\pi/2, \pi/2]$, we obtain the solutions:

$$\begin{array}{lll} y(x) = \sin(x^2+K), & K \in \mathbb{R}, & x \text{ such that } -\frac{\pi}{2} \leq x^2 + K \leq \frac{\pi}{2} \\ y(x) = 1, & x \in \mathbb{R} & \\ y(x) = -1, & x \in \mathbb{R} & \end{array}$$

4. Clearly, the constant solution $y(x) = 1$ satisfies the initial condition. But there is another solution, which can be obtained by imposing that $\sin(0+K) = 1$. This equation is verified for every $K = \frac{\pi}{2} + 2h\pi$. Let us take into account the condition $-\frac{\pi}{2} \leq x^2 + K \leq \frac{\pi}{2}$ and the fact that the solution should be defined in the largest possible interval containing the point $x = 0$: so we can find also the solution $y(x) = \sin(x^2 + \frac{\pi}{2})$.

   In this case, we find two distinct solutions of the Cauchy problem, and in fact, at $y = 1$ the Theorem of existence and uniqueness of solutions does not hold true.

Exercise 11. We have to determine all continuous functions, with continuous first derivative in $\mathbb{R}$, which solve the two equations in the indicated intervals.

We note that, in order to find the solutions of the first equation, it is enough to integrate:

$$y_1(x) = \int x\log(1+x^2)dx = \frac{1}{2}\Big((1+x^2)\log(1+x^2) - (1+x^2)\Big) + C, \qquad C \in \mathbb{R}.$$

The second equation, instead, is a linear one, and there the functions of x are defined for all $x \neq 0$. In particular, all solutions are defined as $x > 1$. Using the formula, we get:

$$\begin{aligned} y_2(x) &= e^{\log|x|}\int -\frac{1}{|x|}\cdot\frac{3x+2}{x^2}\,dx = -x\int\frac{3x+2}{x^3}\,dx \\ &= -x\left(-\frac{3}{x} - \frac{1}{x^2} + C\right) = 3 + \frac{1}{x} + Kx, \qquad K \in \mathbb{R}. \end{aligned}$$

The two functions that we have found should be continuous and differentiable at $x = 1$:

$$\begin{aligned} \lim_{x\to 1^-} y_1(x) &= \lim_{x\to 1^+} y_2(x) \\ \lim_{x\to 1^-} y_1'(x) &= \lim_{x\to 1^+} y_2'(x) \end{aligned}$$

From the first condition we obtain $C = K - \log 2 + 5$. The second one implies

$$\begin{aligned} \lim_{x\to 1^-} y_1'(x) &= \lim_{x\to 1^-} x\log(1+x^2) = \log 2 \\ \lim_{x\to 1^+} y_2'(x) &= \lim_{x\to 1^+} -\tfrac{1}{x^2} + K = K - 1 \end{aligned}$$

and hence $K = \log 2 + 1$ and $C = 6$.

In conclusion, there exists a unique function satisfying the given conditions, which is:

$$y(x) = \begin{cases} \frac{1}{2}\left((1+x^2)\log(1+x^2) - (1+x^2)\right) + 6, & x < 1 \\ 3 + \frac{1}{x} + (\log 2 + 1)x, & x \geq 1 \end{cases}$$

Exercise 12. The equation can be written also as:

$$y' = -\frac{1}{t-1}y + \frac{2t}{t-1}.$$

Hence, it is linear, with $a(t) = -\frac{1}{t-1}$ and $b(t) = \frac{2t}{t-1}$.

1. The two functions $a(t)$ and $b(t)$ are continuous in $(-\infty, 1)$ and $(1, +\infty)$. Since the equation is linear, all solutions are global, which means that they are defined in the whole interval $(-\infty, 1)$ or in $(1, +\infty)$.

2. Using the known formula for linear equations we get:

$$\begin{aligned} y(t) &= e^{-\log|t-1|} \int e^{\log|t-1|} \frac{2t}{t-1} dt \\ &= \frac{1}{|t-1|} \int |t-1| \frac{2t}{t-1} dt \\ &= \frac{\operatorname{sgn}(t-1)}{t-1} \int [\operatorname{sgn}(t-1)](t-1) \frac{2t}{t-1} dt \\ &= \frac{1}{t-1} \int 2t dt = \frac{1}{t-1}(t^2 + C), \qquad C \in \mathbb{R}. \end{aligned}$$

3. We have to find the solution satisfying the initial condition $y(0) = 1$. We find that $y(0) = \frac{1}{-1}C = 1$. Hence, the solution is:

$$y(t) = \frac{1}{t-1}(t^2 - 1) = t + 1, \qquad t \in (-\infty, 1).$$

   The domain of definition of the solution is a crucial issue, since $y(t)$ is defined all over $\mathbb{R}$ as a function, but not as a solution of the given equation: this because its coefficients a, and b are not defined at $t = 1$.

4. We have to find the solution satisfying the initial condition $y(2) = 1$. We find that $y(2) = \frac{1}{1}(4 + C) = 1$. Hence, the solution is:

$$y(t) = \frac{1}{t-1}(t^2 - 3), \qquad t \in (1, +\infty).$$

Exercise 13. The equation is linear of the second order, with constant coefficients.

(a) The equation is homogeneous, its characteristic equation is $z^2 + 3z - 10 = (x-2)(x+5) = 0$. The general integral is:

$$y(t) = c_1 e^2 t + c_2 e^{-5t}, \qquad c_1, c_2 \in \mathbb{R}.$$

(b) We have to look for a particular solution of the form $y(t) = k$, con $k \in \mathbb{R}$. Since $y'(t) = y''(t) = 0$, by substitution we get $-10k = 5$; so, a particular solution of the non homogeneous equation is $y(t) = -\frac{1}{2}$ and its general integral is

$$y(t) = c_1 e^2 t + c_2 e^{-5t} - \frac{1}{2}, \qquad c_1, c_2 \in \mathbb{R}.$$

(c) We should look for a particular solution of the type $y(t) = a + bt$, with $a, b \in \mathbb{R}$. Since $y'(t) = b$ and $y''(t) = 0$, by substitution we get $3b - 10(a + bt) = 3 + 2t$, from which it follows that $a = -\frac{9}{25}$ and $b = -\frac{1}{5}$.

The general integral of the equation is:

$$y(t) = c_1 e^2 t + c_2 e^{-5t} - \frac{9}{25} - \frac{1}{5}t, \qquad c_1, c_2 \in \mathbb{R}.$$

(d) We should look for a particular solution of the type $y(t) = ke^t$, with $k \in \mathbb{R}$. Since $y'(t) = ke^t$ and $y''(t) = ke^t$, replacing them in the equation we have $ke^t + 3ke^t - 10ke^t = 3e^t$, from which we obtain that $k = -\frac{1}{2}$.

The general integral of the equation is:

$$y(t) = c_1 e^2 t + c_2 e^{-5t} - \frac{1}{2}e^t, \qquad c_1, c_2 \in \mathbb{R}.$$

(e) In this case -5 is a root of the characteristic polynomial; the resonance phenomenon occurs, so we have to look for a particular solution of the type $y(t) = kte^{-5t}$, with $k \in \mathbb{R}$. Since $y'(t) = (k - 5kt)e^{-5t}$ and $y''(t) = (-10k + 25kt)e^{-5t}$, replacing them in the equation we get $(-10k + 25kt)e^{-5t} + 3(k - 5kt)e^{-5t} - 10kte^{-5t} = 2e^{-5t}$ from which we obtain $k = -\frac{2}{7}$.

The general integral of the equation is:

$$y(t) = c_1 e^2 t + c_2 e^{-5t} - \frac{2}{7}te^{-5t}, \qquad c_1, c_2 \in \mathbb{R}.$$

(f) In this case $f(t)$ is the sum of the terms appearing in the right-hand sides of the preceding two equations. By the superposition principle a particular solution is the sum of the ones corresponding to the preceding two equations. Hence the general integral is:

$$y(t) = c_1 e^2 t + c_2 e^{-5t} - \frac{1}{2}e^t - \frac{2}{7}te^{-5t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Exercise 14. This equation is linear, of the second order, with constant coefficients.

(a) The equation is homogeneous, its characteristic equation being
$z^2 + 9z = 0$. Its roots are $\pm 3i$, so to them we can associate the two complex solutions e^{3it} e e^{-3it}. By a linear combination with complex coefficients, we obtain the two real solutions $\cos 3t$ e $\sin 3t$, that allows us to construct the general integral:

$$y(t) = c_1 \cos 3t + c_2 \cos 3t, \qquad c_1, c_2 \in \mathbb{R}.$$

(*b*) The value 0 is not a root of the characteristic equation, so we have to look for a particular solution in a polynomial form with degree 1, i.e. $y(t) = at + b$. Since $y' = a$, $y'' = 0$, inserting them in the equation we get that $a = \frac{4}{9}$, $b = 0$. Hence, the general integral is:

$$y(t) = c_1 \cos 3t + c_2 \cos 3t + \frac{4}{9}t, \qquad c_1, c_2 \in \mathbb{R}.$$

(*c*) The value 0 is not a root of the characteristic equation, hence we have to look for a particular solution of the type $y(t) = ke^{3t}$. Since $y'(t) = 3ke^{3t}$ and $y''(t) = 9ke^{3t}$, replacing them in the equation we get $k = \frac{1}{18}$. The general integral is:

$$y(t) = c_1 \cos 3t + c_2 \cos 3t + \frac{1}{18}e^{3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

(*d*) The value $3i$ is not a solution of the characteristic equation, so we should look for a particular solution in the form $y(t) = t(k \cos 3t + h \sin 3t)$. Upon derivation we get:

$$\begin{aligned}
&y(t) = t(k \cos 3t + h \sin 3t) \\
&y'(t) = k \cos 3t + h \sin 3t + t(3h \cos 3t - 3k \sin 3t) \\
&y''(t) = 6(h \cos 3t - k \sin 3t) - 9t(k \cos 3t + h \sin 3t)
\end{aligned}$$

By replacing them in the equation we obtain

$$-6k \sin 3t + 6h \cos 3t = 4 \cos 3t,$$

from which we obtain that $k = 0$ and $h = \frac{2}{3}$. Hence, the general integral is:

$$y(t) = c_1 \cos 3t + c_2 \cos 3t + \frac{2}{3}t \sin 3t, \qquad c_1, c_2 \in \mathbb{R}.$$

(*e*) The value $3 + 3i$ is not a solution of the characteristic equation, so we have to look for a particular solution of the form $y(t) = e^{3t}(k \cos 3t + h \sin 3t)$. Upon derivation we get:

$$\begin{aligned}
&y(t) = 3e^{3t}(k \cos 3t + (h - k) \sin 3t] \\
&y'(t) = 3e^{3t}[(k + h)k \cos 3t + (h - k) \sin 3t) \\
&y''(t) = 9e^{3t}(2h \cos 3t - 2k \sin 3t)
\end{aligned}$$

Inserting them in the equation we get

$$e^{3t}[9(2h + k) \cos 3t + 9(-2k + h) \sin 3t) = 4e^{3t} \cos 3t$$

from which we obtain the linear system

$$\begin{cases} 9(2h + k) = 4 \\ 9(-2k + h) = 0 \end{cases} \iff \begin{cases} k = \frac{4}{45} \\ h = \frac{8}{45} \end{cases}$$

Hence, the general integral is:

$$y(t) = c_1 \cos 3t + c_2 \cos 3t + e^{3t}\left(\frac{4}{45} \cos 3t + \frac{8}{45} \sin 3t\right), \qquad c_1, c_2 \in \mathbb{R}.$$

(f) In this case $f(t) = 4\cos 3t + e^{3t}$ is the sum of the forcing term of case (c) and the one of case (d), so to find a particular solution it suffices to apply the superposition principle, summing up the particular solutions found in those cases. Hence, the general integral is:

$$y(t) = c_1 \cos 3t + c_2 \cos 3t + \frac{1}{18}e^{3t} + \frac{2}{3}t \sin 3t, \qquad c_1, c_2 \in \mathbb{R}.$$

Exercise 15.

(a)

1. The equation is homogeneous, with characteristic equation $z^2 + 4z + 5 = 0$. Its roots are $-2 \pm i$. The general integral is then:

$$y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t), \qquad c_1, c_2 \in \mathbb{R}.$$

2. There are no bounded solutions (except from the zero solution), since all solutions are unbounded in a neighbourhood of $-\infty$.
3. All solutions are bounded in $(0, +\infty)$.
4. All solutions tend to 0 as $t \to +\infty$, since they are the product of an infinitesimal function times a bounded one.

(b)

1. The value i is not a solution of the characteristic equation, so that we have to look for a particular solution of the type $y(t) = k \cos t + h \sin t$. Proceeding as in the preceding exercise, we find that $k = -\frac{1}{8}$ and $h = \frac{1}{8}$. The general integral is:

$$y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) - \frac{1}{8} \cos t + \frac{1}{8} \sin t, \qquad c_1, c_2 \in \mathbb{R}.$$

2. The only bounded solution in $\mathbb{R}$ is the particular solution we have found, which corresponds to $c_1 = c_2 = 0$.
3. All the solutions are bounded in $(0, +\infty)$.
4. None of the solutions tend to 0 as $t \to +\infty$, since the particular solution is bounded, but it does not have the limit as $t \to +\infty$.

(c)

1. The value $-2 + i$ is a solution of the characteristic equation, so we have to look for a particular solution of the type $y(t) = te^{-2t}(k \cos t + h \sin t)$. Proceeding as in the preceding exercise, we find that $k = -\frac{1}{2}$ e $h = 0$. The general integral is then:

$$y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) - \frac{1}{2}te^{-2t} \cos t, \qquad c_1, c_2 \in \mathbb{R}.$$

   Let us note that, in this case, the resonance phenomenon occurs.
2. There are no bounded solutions in $\mathbb{R}$.
3. There are no bounded solutions in $(0, +\infty)$: the forcing term determines a resonance effect, which amplifies the solutions which were bounded when the forcing term was 0.
4. None of the solutions tends to 0 as $t \to +\infty$, for the same reason as before.

Exercise 16. We have to apply the definition of solution in all the three cases.

1. Let us insert $y_1(t) + y_2(t)$ in the equation:

$$\begin{aligned}&(y_1(t) + y_2(t))'' + a(y_1(t) + y_2(t))' + b(y_1(t) + y_2(t)) =\\&(y_1''(t) + ay_1'(t) + by_1(t)) + (y_2''(t) + ay_2'(t) + by_2(t)) =\\&f(t) + f(t) = 2f(t), \qquad \forall t \in I,\end{aligned}$$

since they are both solutions of the same equation.

2. In this case we may proceed analogously:

$$\begin{aligned}&(y_1(t) - y_2(t))'' + a(y_1(t) - y_2(t))' + b(y_1(t) - y_2(t)) =\\&(y_1''(t) + ay_1'(t) + by_1(t)) - (y_2''(t) + ay_2'(t) + by_2(t)) =\\&f(t) - f(t) = 0, \qquad \forall t \in I.\end{aligned}$$

Hence $y_1(t) - y_2(t)$ is solution of the homogeneous equation.

3.
$$\begin{aligned}&(y_1(t) + y_3(t))'' + a(y_1(t) + y_3(t))' + b(y_1(t) + y_3(t)) =\\&(y_1''(t) + ay_1'(t) + by_1(t)) + (y_3''(t) + ay_3'(t) + by_3(t)) =\\&f(t) + g(t), \qquad \forall t \in I,\end{aligned}$$

and hence $y_1 + y_3$ is solution of the given equation.

Exercise 17.

1. $\boxed{\text{False}}$ In fact the characteristic polynomial associated to the homogeneous equation $z^2 + 4z - 5 = 0$ has roots $z = -5$ and $z = 1$. So the general integral of the homogeneous equation is $c_1e^t + c_2e^{-5t}$.

2. $\boxed{\text{False}}$ The functions Ke^{-5t} are already solutions of the homogeneous equation.

   Instead, there exists a solution Kte^{-5t}, for some $K \neq 0$ to be determined.

3. $\boxed{\text{True}}$ From the general theory we know that, since 0 is not a root of the characteristic polynomial, we can look for a polynomial solution of the same degree as $b(t)$.

4. $\boxed{\text{True}}$ In fact, if $y(t) = -\frac{3}{5}$, $y'(t) = y''(t) = 0$, so $0 + 4 \cdot 0 - 5 \cdot \left(-\frac{3}{5}\right) = 3$ is satisfied for every $t \in \mathbb{R}$.