# MATH 105 921 Solutions to Integration Exercises

1) $\int \frac{s^2+1}{s^2-1}\,ds$

**Solution:** Performing polynomial long division, we have that:

$$\begin{aligned}\int \frac{s^2+1}{s^2-1}\,ds &= \int (1+\frac{2}{s^2-1})\,ds\\ &= \int ds + \int \frac{2}{s^2-1}\,ds\\ &= s + \int \frac{2}{s^2-1}\,ds\end{aligned}$$

Using partial fraction on the remaining integral, we get:

$$\frac{2}{s^2-1} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{A(s+1)+B(s-1)}{(s+1)(s-1)} = \frac{(A+B)s+(A-B)}{s^2-1}$$

Thus, $A+B=0$ and $A-B=2$. Adding the two equations together yields $2A=2$, that is, $A=1$, and $B=-1$. So, we have that:

$$\int \frac{2}{s^2-1}\,ds = \int \frac{1}{s-1}\,ds - \int \frac{1}{s+1}\,ds$$

Therefore,

$$\begin{aligned}\int \frac{s^2+1}{s^2-1}\,ds &= s + \int \frac{2}{s^2-1}\,ds\\ &= s + \int \frac{1}{s-1}\,ds - \int \frac{1}{s+1}\,ds\\ &= s + \ln|s-1| - \ln|s+1| + C\end{aligned}$$

2) $\int_4^0 x\sqrt{1+2x}\,dx$

**Solution:** Using direct substitution with $u=1+2x$ and $du=2dx$, we may write

$x = \frac{1}{2}(u-1)$. Moreover, when $x = 4$, $u = 9$, and when $x = 0$, $u = 1$. Thus,

$$\begin{aligned}\int_4^0 x\sqrt{1+2x}\,dx &= \int_9^1 \frac{1}{4}(u-1)\sqrt{u}\,du \\ &= \int_9^1 \frac{1}{4}(u^{\frac{3}{2}} - u^{\frac{1}{2}})\,du \\ &= (\frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}})\,\Big|_9^1 \\ &= (\frac{1}{10} - \frac{1}{6}) - (\frac{243}{10} - \frac{27}{6}) \\ &= \frac{-298}{15}\end{aligned}$$

3) $\int \sin^2 x \cos^2 x\,dx$

**Solution:** Using half-angle identities $\sin^2 x = \frac{1-\cos(2x)}{2}$ and $\cos^2 x = \frac{1+\cos(2x)}{2}$, we get:

$$\begin{aligned}\int \sin^2 x \cos^2 x\,dx &= \int \frac{1}{4}(1-\cos(2x))(1+\cos(2x))\,dx \\ &= \int \frac{1}{4}(1-\cos^2(2x))\,dx \\ &= \int \frac{1}{4}\,dx - \int \frac{1}{4}\cos^2(2x)\,dx \\ &= \frac{x}{4} - \frac{1}{4}\int \cos^2(2x)\,dx\end{aligned}$$

On the remaining integral, we apply the half-angle identity $\cos^2(2x) = \frac{1+\cos(4x)}{2}$, and obtain:

$$\int \cos^2(2x)\,dx = \int \frac{1+\cos(4x)}{2}\,dx = \frac{x}{2} + \frac{1}{8}\sin(4x) + C$$

Hence,

$$\int \sin^2 x \cos^2 x\,dx = \frac{x}{4} - \frac{1}{4}(\frac{x}{2} + \frac{1}{8}\sin(4x)) + C = \frac{x}{8} - \frac{1}{32}\sin(4x) + C$$

4) $\int \sin(\sqrt{w})\,dw$

**Solution:** Using direct substitution with $t = \sqrt{w}$, and $dt = \frac{1}{2\sqrt{w}} dw$, that is, $dw = 2\sqrt{w}\, dt = 2t\, dt$, we get:

$$\int \sin(\sqrt{w})\, dw = \int 2t \sin t\, dt$$

Using integration by part method with $u = 2t$ and $dv = \sin t\, dt$, so $du = 2\, dt$ and $v = -\cos t$, we get:

$$\int 2t \sin t\, dt = -2t\cos t + \int 2\cos t\, dt = -2t\cos t + 2\sin t + C$$

Therefore,

$$\int \sin(\sqrt{w})\, dw = -2\sqrt{w}\cos(\sqrt{w}) + 2\sin(\sqrt{w}) + C$$

5) $\displaystyle\int \frac{\ln(x)}{x}\, dx$

**Solution:** Using direct substitution with $u = \ln(x)$ and $du = \frac{1}{x}\, dx$, we get:

$$\int \frac{\ln(x)}{x}\, dx = \int u\, du = \frac{u^2}{2} + C$$
$$\Rightarrow \int \frac{\ln(x)}{x}\, dx = \frac{1}{2}(\ln(x))^2 + C$$

6) $\displaystyle\int \sin t \cos(2t)\, dt$

**Solution:** Recall the double-angle formula that $\cos(2t) = 2\cos^2 t - 1$, we get:

$$\begin{aligned}\int \sin t\cos(2t)\, dt &= \int \sin t(2\cos^2 t - 1)\, dt \\ &= \int 2\sin t\cos^2 t\, dt - \int \sin t\, dt = \int 2\sin t \cos^2 t\, dt + \cos t\end{aligned}$$

On the remaining integral, using direct substitution with $u = \cos t$ and $du = -\sin t\, dt$, we have that:

$$\int 2\sin t\cos^2 t\, dt = \int -2u^2\, du = -\frac{2}{3}u^3 + C = -\frac{2}{3}\cos^3 t + C$$

Therefore,

$$\int \sin t \cos(2t)\, dt = -\frac{2}{3}\cos^3 t + \cos t + C$$

7) $\displaystyle\int \frac{x+1}{4+x^2}\, dx$

**Solution:** Observe that we may split the integral as follows:

$$\int \frac{x+1}{4+x^2}\, dx = \int \frac{x}{4+x^2}\, dx + \int \frac{1}{4+x^2}\, dx$$

On the first integral on the right hand side, we use direct substitution with $u = 4+x^2$, and $du = 2x\, dx$. We get:

$$\int \frac{x}{4+x^2}\, dx = \int \frac{1}{2u}\, du = \ln|2u| + C = \ln(8+2x^2) + C$$

On the second integral on the right hand side, we use inverse trigonometric substitution with $2\tan t = x$ (or equivalently, $t = \arctan\left(\frac{x}{2}\right)$), so $2\sec^2 t\, dt = dx$. Thus,

$$\begin{aligned}\int \frac{1}{4+x^2}\, dx &= \int \frac{1}{4+4\tan^2 t} 2\sec^2 t\, dt = \int \frac{2\sec^2 t}{4\sec^2 t}\, dt \\ &= \int \frac{1}{2}\, dt = \frac{t}{2} + C = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C\end{aligned}$$

Therefore,

$$\int \frac{x+1}{4+x^2}\, dx = \int \frac{x}{4+x^2}\, dx + \int \frac{1}{4+x^2}\, dx \quad = \ln(8+2x^2) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

8) $\displaystyle\int \frac{\sin(\tan\theta)}{\cos^2\theta}\, d\theta$

**Solution:** Using direct substitution with $u = \tan\theta$ and $du = \sec^2\theta\, d\theta$, we get:

$$\begin{aligned}&\int \frac{\sin(\tan\theta)}{\cos^2\theta}\, d\theta = \int \sec^2\theta \sin(\tan\theta)\, d\theta = \int \sin u\, du = -\cos u + C \\ \Rightarrow &\int \frac{\sin(\tan\theta)}{\cos^2\theta}\, d\theta = -\cos(\tan\theta) + C\end{aligned}$$

9) $\displaystyle\int x\sqrt{3-2x-x^2}\,dx$

**Solution:** Completing the square, we get $3-2x-x^2 = 4-(x+1)^2$. Using direct substitution with $u = x+1$ and $du = dx$, we get:

$$\int x\sqrt{3-2x-x^2}\,dx = \int (u-1)\sqrt{4-u^2}\,du = \int u\sqrt{4-u^2}\,du - \int \sqrt{4-u^2}\,du$$

For the first integral on the right hand side, using direct substitution with $t = 4-u^2$, and $dt = -2u\,du$, we get:

$$\int u\sqrt{4-u^2}\,du = \int -\frac{1}{2}\sqrt{t}\,dt = -\frac{1}{3}t^{\frac{3}{2}} + C = -\frac{1}{3}(4-u^2)^{\frac{3}{2}} + C$$

For the second integral on the right hand side, using inverse trigonometric substitution with $2\sin s = u$, that is, $s = \arcsin\left(\frac{u}{2}\right)$, and $2\cos s\,ds = du$, we get:

$$\begin{aligned}
\int \sqrt{4-u^2}\,du &= \int \sqrt{4-4\sin^2 s}\,2\cos s\,ds = \int 4\cos^2 s\,ds \\
&= \int (2+2\cos(2s))\,ds \qquad \text{(using half-angle formula } \cos^2 s = \frac{1+\cos(2s)}{2}\text{)} \\
&= 2s + \sin(2s) + C \\
&= 2s + 2\sin s\cos s + C \qquad \text{(using double-angle formula } \sin(2s) = 2\sin s\cos s\text{)} \\
&= 2\arcsin\left(\frac{u}{2}\right) + 2\sin\left(\arcsin\left(\frac{u}{2}\right)\right)\cos\left(\arcsin\left(\frac{u}{2}\right)\right) + C \\
&= 2\arcsin\left(\frac{u}{2}\right) + u\left(\frac{\sqrt{4-u^2}}{2}\right) + C
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int x\sqrt{3-2x-x^2}\,dx &= \int u\sqrt{4-u^2}\,du - \int \sqrt{4-u^2}\,du \\
&= -\frac{1}{3}(4-u^2)^{\frac{3}{2}} - 2\arcsin\left(\frac{u}{2}\right) - u\left(\frac{\sqrt{4-u^2}}{2}\right) + C
\end{aligned}$$

$$\Rightarrow \int x\sqrt{3-2x-x^2}\,dx = -\frac{1}{3}(4-(x+1)^2)^{\frac{3}{2}} - 2\arcsin\left(\frac{x+1}{2}\right) - (x+1)\left(\frac{\sqrt{4-(x+1)^2}}{2}\right) + C$$

10) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 z\cos z\,dz$

**Solution:** Using direct substitution with $u = \sin z$, and $du = \cos z\, dz$, when $z = 0$, then $u = 0$, and when $z = \frac{\pi}{3}$, $u = \frac{\sqrt{3}}{2}$. We have that:

$$\int_0^{\frac{\pi}{3}} \sin^3 z \cos z\, dz = \int_0^{\frac{\sqrt{3}}{2}} u^3\, du = \frac{u^4}{4} \Big|_0^{\frac{\sqrt{3}}{2}} = \frac{9}{64}$$
$$\Rightarrow \int_0^{\frac{\pi}{3}} \sin^3 z \cos z\, dz = \frac{9}{64}$$

11) $\displaystyle\int \frac{1}{3x^2 + 2x + 1}\, dx$

**Solution:** Completing the square, we get that $3x^2 + 2x + 1 = 3\left(x + \frac{1}{3}\right)^2 + \frac{2}{3} = \frac{2}{3}\left(\frac{9}{2}\left(x + \frac{1}{3}\right)^2 + 1\right)$. Using direct substitution with $u = \frac{3}{\sqrt{2}}\left(x + \frac{1}{3}\right)$, and $du = \frac{3}{\sqrt{2}}\, dx$, we get:

$$\int \frac{1}{3x^2 + 2x + 1}\, dx = \int \frac{3}{2(\frac{9}{2}(x + \frac{1}{3})^2 + 1)}\, dx = \int \frac{1}{\sqrt{2}(u^2 + 1)}\, du = \frac{1}{\sqrt{2}} \arctan u + C$$
$$\Rightarrow \int \frac{1}{3x^2 + 2x + 1}\, dx = \frac{1}{\sqrt{2}} \arctan\left(\frac{3}{\sqrt{2}}\left(x + \frac{1}{3}\right)\right) + C$$

12) $\displaystyle\int \frac{1}{e^t + 1}\, dt$

**Solution:** Using direct substitution with $u = e^t + 1$ and $du = e^t\, dt$, so $dt = \frac{1}{e^t}\, du = \frac{1}{u - 1}\, du$. Hence, we get:

$$\int \frac{1}{e^t + 1}\, dt = \int \frac{1}{u(u - 1)}\, du$$

Using partial fraction, we get:

$$\frac{1}{u(u - 1)} = \frac{A}{u} + \frac{B}{u - 1} = \frac{A(u - 1) + Bu}{u(u - 1)} = \frac{(A + B)s + (-A)}{u(u - 1)}$$

Thus, $A + B = 0$ and $-A = 1$. So, $A = -1$, and $B = 1$. Thus, we have that:

$$\int \frac{1}{u(u-1)}\,du = \int \frac{-1}{u}\,du + \int \frac{1}{u-1}\,du$$

Therefore,

$$\int \frac{1}{u(u-1)}\,du = \int \frac{-1}{u}\,du + \int \frac{1}{u-1}\,du = -\ln|u| + \ln|u-1| + C$$
$$\Rightarrow \int \frac{1}{e^t+1}\,dt = -\ln|e^t+1| + \ln|e^t| + C = -\ln|e^t+1| + t + C$$

13) $\int e^{3a}\cos(3a)\,da$

**Solution:** Using direct substitution with $t = 3a$, and $dt = 3\,da$, we get:

$$\int e^{3a}\cos(3a)\,da = \int \frac{1}{3}e^t\cos t\,dt$$

Using integration by parts with $u = \cos t$, $du = -\sin t\,dt$, and $dv = e^t\,dt$, $v = e^t$, we get:

$$\int \frac{1}{3}e^t\cos t\,dt = \frac{1}{3}e^t\cos t + \frac{1}{3}\int e^t\sin t\,dt$$

Using integration by parts again on the remaining integral with $u_1 = \sin t$, $du_1 = \cos t\,dt$, and $dv_1 = e^t\,dt$, $v_1 = e^t$, we get:

$$\frac{1}{3}\int e^t\sin t\,dt = \frac{1}{3}\sin t e^t - \frac{1}{3}\int e^t\cos t\,dt$$

Thus,

$$\int \frac{1}{3}e^t\cos t\,dt = \frac{1}{3}e^t\cos t + \frac{1}{3}\sin t e^t - \frac{1}{3}\int e^t\cos t\,dt$$
$$\Rightarrow \int \frac{1}{3}e^t\cos t\,dt = \frac{1}{6}e^t\cos t + \frac{1}{6}e^t\sin t + C$$

Therefore,

$$\int e^{3a}\cos(3a)\,da = \frac{1}{6}e^{3a}\cos(3a) + \frac{1}{6}e^{3a}\sin(3a) + C$$

14) $\displaystyle\int \frac{x^2}{1+x^6}\,dx$

**Solution:** Using direct substitution with $u = x^3$, and $du = 3x^2\,dx$, we get:

$$\int \frac{x^2}{1+x^6}\,dx = \int \frac{1}{3(1+u^2)}\,du = \frac{1}{3}\arctan u + C = \frac{1}{3}\arctan(x^3) + C$$

15) $\displaystyle\int \frac{1}{t(\ln t)^2}\,dt$

**Solution:** Using direct substitution with $u = \ln(t)$ and $du = \frac{1}{t}\,dt$, we get:

$$\int \frac{1}{t(\ln t)^2}\,dt = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C$$

$$\Rightarrow \int \frac{1}{t(\ln t)^2}\,dt = -\frac{1}{\ln t} + C$$

16) $\displaystyle\int \frac{xe^{2x}}{(2x+1)^2}\,dx$

**Solution:** Using integration by parts with $u = xe^{2x}$, $du = (e^{2x} + 2xe^{2x})\,dx$, and $dv = (2x+1)^{-2}\,dx$, $v = -\frac{1}{2(2x+1)}$, we get:

$$\int \frac{xe^{2x}}{(2x+1)^2}\,dx = -\frac{xe^{2x}}{2(2x+1)} + \int \frac{e^{2x} + 2xe^{2x}}{2(2x+1)}\,dx$$

On the remaining integral, using direct substitution with $u = 2x+1$, and $du = 2\,dx$, we get:

$$\int \frac{e^{2x} + 2xe^{2x}}{2(2x+1)}\,dx = \int \frac{e^{u-1} + (u-1)e^{u-1}}{4u}\,du = \int \frac{1}{4}e^{u-1}\,du = \frac{1}{4}e^{u-1} + C = \frac{1}{4}e^{2x} + C$$

Therefore,

$$\int \frac{xe^{2x}}{(2x+1)^2}\,dx = -\frac{xe^{2x}}{2(2x+1)} + \frac{1}{4}e^{2x} + C = \frac{e^{2x}}{4(2x+1)} + C$$

17) $\displaystyle\int (\tan x + \cot x)^2\, dx$

**Solution:**

$$\begin{aligned}\int (\tan x + \cot x)^2\, dx &= \int (\tan^2 x + 2\tan x \cot x + \cot^2 x)\, dx\\ &= \int (\sec^2 x - 1 + 2 + \csc^2 x - 1)\, dx \qquad \text{(using identities for } \tan^2 x \text{ and } \cot^2 x\text{)}\\ &= \int (\sec^2 x + \csc^2 x)\, dx\\ &= \tan x - \cot x + C\end{aligned}$$

18) $\displaystyle\int te^{t^2} \sin(t^2)\, dt$

**Solution:** Using direct substitution with $x = t^2$ and $dx = 2t\, dt$, we get:

$$\int te^{t^2} \sin(t^2)\, dt = \frac{1}{2}\int e^x \sin x\, dx$$

Using integration by parts with $u = \sin x$, $du = \cos x\, dx$, and $dv = e^x\, dx$, $v = e^x$, we get:

$$\int \frac{1}{2} e^x \sin x\, dx = \frac{1}{2} e^x \sin x - \frac{1}{2}\int e^x \cos x\, dx$$

Using integration by parts again on the remaining integral with $u_1 = \cos x$, $du_1 = -\sin x\, dx$, and $dv_1 = e^x\, dx$, $v_1 = e^x$, we get:

$$\frac{1}{2}\int e^x \cos x\, dx = \frac{1}{2} e^x \cos x + \frac{1}{2}\int e^x \sin x\, dx$$

Thus,

$$\begin{aligned}\int \frac{1}{2} e^x \sin x\, dx &= \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x - \frac{1}{2}\int e^x \sin x\, dx\\ \Rightarrow \int \frac{1}{2} e^x \sin x\, dx &= \frac{1}{4} e^x \sin x - \frac{1}{4} e^x \cos x + C\end{aligned}$$

Therefore,

$$\int te^{t^2} \sin(t^2)\, dt = \frac{1}{4} e^{t^2} \sin(t^2) - \frac{1}{4} e^{t^2} \cos(t^2) + C$$

19) $\displaystyle\int \frac{2p-4}{p^2-p}\,dp$

**Solution:** Using partial fraction, we get:

$$\frac{2p-4}{p(p-1)} = \frac{A}{p} + \frac{B}{p-1} = \frac{A(p-1)+Bp}{p(p-1)} = \frac{(A+B)p+(-A)}{p(p-1)}$$

Thus, $A+B=2$ and $-A=-4$. So, $A=4$, and $B=-2$. We have that:

$$\int \frac{2p-4}{p(p-1)}\,dp = \int \frac{4}{p}\,dp - \int \frac{2}{p-1}\,dp$$
$$\Rightarrow \int \frac{2p-4}{p(p-1)}\,dp = 4\ln|p| - 2\ln|p-1| + C$$

20) $\displaystyle\int_3^4 \frac{1}{(3x-7)^2}\,dx$

**Solution:** Using direct substitution with $u=3x-7$, and $du=3\,dx$, when $x=3$, then $u=2$, and when $x=4$, $u=5$. We have that:

$$\int_3^4 \frac{1}{(3x-7)^2}\,dx = \int_2^5 \frac{1}{3u^2}\,du = \frac{-1}{3u}\,\Big|_2^5 = -\frac{1}{15}+\frac{1}{6} = \frac{1}{10}$$
$$\Rightarrow \int_3^4 \frac{1}{(3x-7)^2}\,dx = \frac{1}{10}$$

21) $\displaystyle\int \frac{t^3}{(2-t^2)^{\frac{5}{2}}}\,dt$

**Solution:** Using direct substitution with $u=2-t^2$, and $du=-2t\,dt$, we get:

$$\int \frac{t^3}{(2-t^2)^{\frac{5}{2}}}\,dt = \int \frac{t^2}{(2-t^2)^{\frac{5}{2}}}\,(t\,dt) = \int -\frac{2-u}{2u^{\frac{5}{2}}}\,du$$
$$= \int (-u^{-\frac{5}{2}} + \frac{1}{2}u^{-\frac{3}{2}})\,du$$
$$= \frac{2}{3}u^{-\frac{3}{2}} - u^{-\frac{1}{2}} + C$$
$$\Rightarrow \int \frac{t^3}{(2-t^2)^{\frac{5}{2}}}\,dt = \frac{2}{3}(2-t^2)^{-\frac{3}{2}} - (2-t^2)^{-\frac{1}{2}} + C$$

22) $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$

**Solution:** Using inverse trigonometric substitution with $x = 2\sin y$, that is, $y = \arcsin\left(\frac{x}{2}\right)$, and $dx = 2\cos y\,dy$, we get:

$$\begin{aligned}\int \frac{1}{x^2\sqrt{4-x^2}}\,dx &= \int \frac{2\cos y}{4\sin^2 y\sqrt{4-4\sin^2 y}}\,dy = \int \frac{2\cos y}{4\sin^2 y(2\cos y)}\,dy \\ &= \int \frac{1}{4}\csc^2 y\,dy = -\frac{1}{4}\cot y + C\end{aligned}$$

Therefore,

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{1}{4}\cot(\arcsin\left(\frac{x}{2}\right)) + C = -\frac{\sqrt{4-x^2}}{4x} + C$$

23) $\displaystyle\int \sqrt{y^2-1}\,dy$

**Solution:** Using inverse trigonometric substitution with $y = \sec u$, that is, $u = \arccos\left(\frac{1}{y}\right)$, and $dy = \sec u\tan u\,du$, we get:

$$\begin{aligned}\int \sqrt{y^2-1}\,dy &= \int \sqrt{\sec^2 u - 1}(\sec u\tan u\,du) = \int \tan^2 u\sec u\,du \\ &= \int(\sec^2 u - 1)\sec u\,du = \int \sec^3 u\,du - \int \sec u\,du\end{aligned}$$

For the second integral on the right hand side, we have that:

$$\int \sec u\,du = \ln|\sec u + \tan u| + C$$

For the first integral on the right hand side, we use the reduction formula:

$$\int \sec^3 u\,du = \frac{1}{2}\tan u\sec u + \frac{1}{2}\int \sec u\,du = \frac{1}{2}\tan u\sec u + \frac{1}{2}\ln|\sec u + \tan u| + C$$

Observe that since $u = \arccos\left(\frac{1}{y}\right)$, we have that $\tan u = \sqrt{y^2-1}$. Therefore,

$$\begin{aligned}\int \sec^3 u\,du - \int \sec u\,du &= \frac{1}{2}\tan u\sec u - \frac{1}{2}\ln|\sec u + \tan u| + C \\ \Rightarrow \int \sqrt{y^2-1}\,dy &= \frac{1}{2}y\sqrt{y^2-1} - \frac{1}{2}\ln|y + \sqrt{y^2-1}| + C\end{aligned}$$

24) $\displaystyle\int x \sin x \cos x \, dx$

**Solution:** Using the double angle identity $\sin(2x) = 2\sin x \cos x$, we have that:

$$\int x \sin x \cos x \, dx = \frac{1}{2}\int x \sin(2x)\, dx$$

Using direct substitution with $t = 2x$, and $dt = 2\, dx$, we get:

$$\frac{1}{2}\int x \sin(2x)\, dx = \frac{1}{8}\int t \sin t \, dt$$

Using integration by parts with $u = t$, $du = dt$, and $dv = \sin t \, dt$, $v = -\cos t$, we get:

$$\frac{1}{8}\int t \sin t \, dt = -\frac{1}{8} t \cos t + \frac{1}{8}\int \cos t \, dt = -\frac{1}{8} t \cos t + \frac{1}{8}\sin t + C$$

Therefore,

$$\int x \sin x \cos x \, dx = -\frac{1}{4} x \cos(2x) + \frac{1}{8}\sin(2x) + C$$

25) $\displaystyle\int (1 + \cos\theta)^2 \, d\theta$

**Solution:**

$$\begin{aligned}
\int (1+\cos\theta)^2 \, d\theta &= \int (1 + 2\cos\theta + \cos^2\theta)\, d\theta \\
&= \int d\theta + 2\int \cos\theta \, d\theta + \int \cos^2\theta \, d\theta \\
&= \theta + 2\sin\theta + \int \left(\frac{1+\cos(2\theta)}{2}\right) d\theta \qquad \text{(using half-angle formula)} \\
&= \theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C \\
\Rightarrow \int (1+\cos\theta)^2 \, d\theta &= \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin(2\theta) + C
\end{aligned}$$

26) $\displaystyle\int \frac{1}{\sqrt{4x - x^2}}\, dx$

**Solution:** Completing the square yields $4x - x^2 = 4 - (x-2)^2$. Using direct substitution with $u = x - 2$, and $du = dx$, we get:

$$\int \frac{1}{\sqrt{4x - x^2}}\, dx = \int \frac{1}{\sqrt{4 - u^2}}\, du$$

Using inverse trigonometric substitution with $u = 2\sin t$, that is, $t = \arcsin\left(\frac{u}{2}\right)$, and $du = 2\cos t\, dt$, we get:

$$\int \frac{1}{\sqrt{4 - u^2}}\, du = \int \frac{2\cos t}{\sqrt{4 - \sin^2 t}}\, dt = \int \frac{2\cos t}{2\cos t}\, dt = \int dt = t + C$$

$$\Rightarrow \int \frac{1}{\sqrt{4x - x^2}}\, dx = \arcsin\left(\frac{x-2}{2}\right) + C$$

27) $\displaystyle\int_0^1 \frac{1}{1 + x^{\frac{1}{3}}}\, dx$

**Solution:** Using direct substitution with $u = 1 + x^{\frac{1}{3}}$, and $du = \frac{1}{3}x^{\frac{-2}{3}}\, dx$, so $dx = 3x^{\frac{2}{3}}\, du = 3(u-1)^2\, du$. When $x = 0$, $u = 1$ and when $x = 1$, $u = 2$. We have that:

$$\int_0^1 \frac{1}{1 + x^{\frac{1}{3}}}\, dx = \int_1^2 \frac{3(u-1)^2}{u}\, du = \int_1^2 (3u - 6 + \frac{3}{u})\, du$$

$$= (\frac{3}{2}u^2 - 6u + 3\ln|u|)\,\Big|_1^2$$

$$= (6 - 12 + 3\ln 2) - (\frac{3}{2} - 6 + 3\ln 1) = -\frac{3}{2} + 3\ln 2$$

$$\Rightarrow \int_0^1 \frac{1}{1 + x^{\frac{1}{3}}}\, dx = -\frac{3}{2} + 3\ln 2.$$

28) $\displaystyle\int \frac{1}{x^3 + x}\, dx$

**Solution:** Using partial fractions, we have:

$$\frac{1}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} = \frac{A(x^2 + 1) + (Bx + C)x}{x^3 + x} = \frac{(A + B)x^2 + Cx + A}{x^3 + x}$$

So, $A + B = 0$, $C = 0$ and $A = 1$. So, $B = -1$ and we get:

$$\int \frac{1}{x^3 + x}\, dx = \int \frac{1}{x}\, dx - \int \frac{x}{x^2 + 1}\, dx = \ln|x| - \int \frac{x}{x^2 + 1}\, dx$$

On the remaining integral, using direct substitution with $u = x^2 + 1$ and $du = 2x\,dx$, we get:

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{2u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$$

Therefore,

$$\int \frac{1}{x^3+x}\,dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

*Remark:* *This involves partial fractions with non-linear factors, which you are not required to master in this course!*

29) $\int \ln(1+t)\,dt$

**Solution:** Using direct substitution with $s = 1+t$, and $ds = dt$, we have that:

$$\int \ln(1+t)\,dt = \int \ln s\,ds$$

Using integration by parts with $u = \ln s$, $du = \frac{1}{s}\,ds$, and $dv = ds$, $v = s$, we get:

$$\int \ln s\,ds = s\ln s - \int s\frac{1}{s}\,ds = s\ln s - \int\,ds = s\ln s - s + C$$

Therefore,

$$\int \ln(1+t)\,dt = (1+t)\ln(1+t) - (1+t) + C$$

30) $\int \sin(3x)\cos(5x)\,dx$

**Solution:** Using the trigonometric identity that $\sin a\cos b = \frac{1}{2}(\sin(a+b)+\sin(a-b))$, we get:

$$\int \sin(3x)\cos(5x)\,dx = \int \frac{1}{2}(\sin(8x)+\sin(-2x))\,dx = -\frac{1}{16}\cos(8x) + \frac{1}{4}\cos(-2x) + C$$

*Remark:* *You are not required to memorize any sum to product or product to sum trigonometric identities!*

31) $\displaystyle\int \frac{1}{k^2-6k+9}\,dk$

**Solution:** By completing the square, we observe that $k^2-6k+9=(k-3)^2$. So, using direct substitution with $u=k-3$, and $du=dk$, we have that:

$$\int \frac{1}{k^2-6k+9}\,dk = \int \frac{1}{(k-3)^2}\,dk = \int \frac{1}{u^2}\,du = -\frac{1}{u}+C$$
$$\Rightarrow \int \frac{1}{k^2-6k+9}\,dk = -\frac{1}{k-3}+C$$

32) $\displaystyle\int \frac{1}{\sec x-1}\,dx$

**Solution:** Since $\sec x = \dfrac{1}{\cos x}$, we get:

$$\int \frac{1}{\sec x-1}\,dx = \int \frac{\cos x}{1-\cos x}\,dx = \int\left(-1+\frac{1}{1-\cos x}\right)dx = -x+\int\frac{1}{1-\cos x}\,dx$$

For the remaining integral, use a direct substitution with $t=\tan\left(\frac{x}{2}\right)$, so $dt=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$. We also can compute that $\sec\left(\frac{x}{2}\right)=\sqrt{t^2+1}$, $\cos\left(\frac{x}{2}\right)=\frac{1}{\sqrt{t^2+1}}$ and $\sin\left(\frac{x}{2}\right)=\frac{t}{\sqrt{t^2+1}}$. So, $dx=\frac{2}{t^2+1}\,dt$. Using double angle formula, we get:

$$\cos x = \cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right) = \frac{1}{t^2+1}-\frac{t^2}{t^2+1} = \frac{1-t^2}{t^2+1}$$

So, after the substitution, we get:

$$\int \frac{1}{1-\cos x}\,dx = \int \frac{1}{1-\frac{1-t^2}{t^2+1}}\left(\frac{2}{t^2+1}\right)dt = \int \frac{1}{t^2}\,dt$$
$$= -\frac{1}{t}+C = -\cot\left(\frac{x}{2}\right)+C$$

Therefore,

$$\int \frac{1}{\sec x-1}\,dx = -x-\cot\left(\frac{x}{2}\right)+C$$

<u>Remark</u>: *This is an extremely challenging question; do not panic if you do not know how to solve it!*

33) $\displaystyle\int_0^1 \frac{2}{e^{-x}+1}\,dx$

**Solution:** Using direct substitution with $u = e^{-x}+1$, and $du = -e^{-x}\,dx$, that is $dx = -\frac{1}{u-1}\,du$. When $x = 0$, $u = 2$, and when $x = 1$, $u = e^{-1}+1$. So, we get:

$$\int_0^1 \frac{2}{e^{-x}+1}\,dx = \int_2^{e^{-1}+1} \frac{-2}{u(u-1)}\,du$$

Using partial fraction, we get:

$$\frac{-2}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1)+Bu}{u(u-1)} = \frac{(A+B)s+(-A)}{u(u-1)}$$

Thus, $A+B = 0$ and $-A = -2$. So, $A = 2$, and $B = -2$. Thus, we have that:

$$\int_2^{e^{-1}+1} \frac{-2}{u(u-1)}\,du = \int_2^{e^{-1}+1} \frac{2}{u}\,du - \int_2^{e^{-1}+1} \frac{2}{u-1}\,du$$

Therefore,

$$\int_2^{e^{-1}+1} \frac{-2}{u(u-1)}\,du = (2\ln|u| - 2\ln|u-1|)\,|_2^{e^{-1}+1} = (2\ln(e^{-1}+1)+2) - (2\ln 2 - 0)$$

$$\Rightarrow \int \frac{2}{e^{-x}+1}\,dx = 2\ln(e^{-1}+1) + 2 - 2\ln 2.$$

34) $\displaystyle\int \frac{1}{c^2-6c+10}\,dc$

**Solution:** Completing the square yields $c^2-6c+10 = (c-3)^2+1$. So, using direct substitution with $u = c-3$, and $du = dc$, we have that:

$$\int \frac{1}{c^2-6c+10}\,dc = \int \frac{1}{(c-3)^2+1}\,dc = \int \frac{1}{u^2+1}\,du = \arctan u + C$$

$$\Rightarrow \int \frac{1}{c^2-6c+10}\,dc = \arctan(c-3) + C$$

35) $\displaystyle\int f(x)f'(x)\,dx$

**Solution:** Using direct substitution with $u = f(x)$, and $du = f'(x)\,dx$, we get:

$$\int f(x)f'(x)\,dx = \int u\,du = \frac{1}{2}u^2 + C$$
$$\Rightarrow \int f(x)f'(x)\,dx = \frac{1}{2}(f(x))^2 + C$$

36) $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$

**Solution:** Completing the square, we get $x^2 + 4x + 5 = (x + 2)^2 + 1$. Using direct substitution with $u = x + 2$ and $du = dx$, we get:

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{(x+2)^2+1}\,dx = \int \frac{1}{u^2+1}\,du = \arctan(u) + C$$
$$\Rightarrow \int \frac{1}{x^2+4x+5}\,dx = \arctan(x+2) + C$$

37) $\displaystyle\int_0^2 \frac{1}{(3+5x)^2}\,dx$

**Solution:** Using direct substitution with $u = 3 + 5x$, and $du = 5\,dx$, when $x = 0$, then $u = 3$, and when $x = 2$, $u = 13$. We have that:

$$\int_0^2 \frac{1}{(3+5x)^2}\,dx = \int_3^{13} \frac{1}{5u^2}\,du = \frac{-1}{5u}\,\Big|_3^{13} = -\frac{1}{65} + \frac{1}{15} = \frac{2}{39}$$
$$\Rightarrow \int_0^2 \frac{1}{(3+5x)^2}\,dx = \frac{2}{39}$$

38) $\displaystyle\int \sin(\ln u)\,du$

**Solution:** Using direct substitution with $t = \ln u$, that is, $u = e^t$, and $du = e^t\,dt$, we have that:

$$\int \sin(\ln u)\,du = \int e^t \sin t\,dt$$

Using integration by parts twice to compute the integral on the right hand side (see the solution of question 18 for details), we have that:

$$\int e^t \sin t \, dt = \frac{1}{2} e^t \sin t - \frac{1}{2} e^t \cos t + C$$

Therefore,

$$\int \sin(\ln u) \, du = \frac{1}{2} e^{\ln u} \sin(\ln u) - \frac{1}{2} e^{\ln u} \cos(\ln u) + C = \frac{1}{2} u \sin(\ln u) - \frac{1}{2} u \cos(\ln u) + C$$

39) $\int r(\ln r)^2 \, dr$

**Solution:** Using integration by parts with $u = (\ln r)^2$, $du = \frac{2 \ln r}{r} \, dr$, and $dv = r \, dr$, $v = \frac{r^2}{2}$, we get that:

$$\int r(\ln r)^2 \, dr = \frac{r^2 (\ln r)^2}{2} - \int r \ln r \, dr$$

Using integration by parts again on the remaining integral with $u_1 = \ln r$, $du_1 = \frac{1}{r} \, dr$, and $dv_1 = r \, dr$, $v_1 = \frac{r^2}{2}$, we get that:

$$\int r \ln r \, dr = \frac{r^2 \ln r}{2} - \int \frac{r}{2} \, dr = \frac{r^2 \ln r}{2} - \frac{r^2}{4} + C$$

Therefore,

$$\int r(\ln r)^2 \, dr = \frac{r^2 (\ln r)^2}{2} - \frac{r^2 \ln r}{2} + \frac{r^2}{4} + C$$

40) $\int \frac{1}{x^3 - x} \, dx$

**Solution:** Using partial fraction, we get:

$$\frac{1}{x^3 - x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} = \frac{A(x^2-1) + B(x^2-x) + C(x^2+x)}{x^3-x}$$
$$= \frac{(A+B+C)x^2 + (C-B)x + (-A)}{x^3-x}$$

Thus, $A+B+C=0$, $C-B=0$ and $-A=1$. Therefore, $A=-1$, and $B+C=1$, which gives $C=\frac{1}{2}$ and $B=-\frac{1}{2}$. So,

$$\int \frac{1}{x^3-x}\,dx = \int -\frac{1}{x}\,dx - \int \frac{1}{2(x+1)}\,dx + \int \frac{1}{2(x-1)}\,dx$$
$$\Rightarrow \int \frac{1}{x^3-x}\,dx = -\ln|x| - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

<u>Remark:</u> *This involves partial fractions with 3 distinct roots in the denominator, which you are not required to master in this course!*

41) $\int \sec^3 u\,du$

**Solution:** We use the reduction formula:

$$\int \sec^3 u\,du = \frac{1}{2}\tan u \sec u + \frac{1}{2}\int \sec u\,du = \frac{1}{2}\tan u \sec u + \frac{1}{2}\ln|\sec u + \tan u| + C$$

42) $\int \frac{\sqrt{x^2-2x-8}}{x-1}\,dx$

**Solution:** Observe that $x^2-2x-8 = (x-1)^2 - 9$. Using direct substitution with $t = x-1$, and $dt = dx$, we get:

$$\int \frac{\sqrt{x^2-2x-8}}{x-1}\,dx = \int \frac{\sqrt{t^2-9}}{t}\,dt$$

Using inverse trigonometric substitution with $t = 3\sec y$, and $dt = 3\sec y \tan y\,dy$, we get:

$$\begin{aligned}
\int \frac{\sqrt{t^2-9}}{t}\,dt &= \int \frac{\sqrt{9\sec^2 y-9}}{3\sec y} 3\sec y\tan y\,dy = \int 3\tan^2 y\,dy \\
&= \int 3(\sec^2 y - 1)\,dy = 3\tan y - 3y + C \\
\Rightarrow \int \frac{\sqrt{x^2-2x-8}}{x-1}\,dx &= 3\tan(\arccos\left(\frac{3}{t}\right)) - 3\arccos\left(\frac{3}{t}\right) + C = \sqrt{t^2-9} - 3\arccos\left(\frac{3}{t}\right) + C \\
&= \sqrt{(x-1)^2-9} - 3\arccos\left(\frac{3}{x-1}\right) + C
\end{aligned}$$

43) $\displaystyle\int \frac{\sqrt{r^2-1}}{r}\,dr$

**Solution:** Using inverse trigonometric substitution with $\sec s = r$, that is, $s = \arccos\left(\frac{1}{r}\right)$, and $\sec s\tan s\,ds = dr$, we get:

$$\begin{aligned}
\int \frac{\sqrt{r^2-1}}{r}\,dr &= \int \frac{\sqrt{\sec^2 s-1}}{\sec s}\sec\tan s\,ds = \int \tan^2 s\,ds \\
&= \int (\sec^2 s - 1)\,ds = \tan s - s + C \\
\Rightarrow \int \frac{\sqrt{r^2-1}}{r}\,dr &= \tan(\arccos\left(\frac{1}{r}\right)) - \arccos\left(\frac{1}{r}\right) + C = \sqrt{r^2-1} - \arccos\left(\frac{1}{r}\right) + C
\end{aligned}$$

44) $\displaystyle\int (e^{t^2}+16)te^{t^2}\,dt$

**Solution:** Using direct substitution with $u = e^{t^2}$ and $du = 2te^{t^2}\,dt$, we get:

$$\begin{aligned}
\int (e^{t^2}+16)te^{t^2}\,dt &= \int \frac{1}{2}(u+16)\,du = \frac{1}{4}u^2 + 8u + C \\
\Rightarrow \int (e^{t^2}+16)te^{t^2}\,dt &= \frac{1}{4}e^{2t^2} + 8e^{t^2} + C
\end{aligned}$$

45) $\displaystyle\int \sqrt{y}\ln y\,dy$

**Solution:** Using integration by parts with $u = \ln y$, $du = \frac{1}{y}\,dy$ and $dv = \sqrt{y}\,dy$, $v = \frac{2}{3}y^{\frac{3}{2}}$, we get:

$$\int \sqrt{y}\ln y\,dy = \frac{2}{3}y^{\frac{3}{2}}\ln y - \int \frac{2}{3}y^{\frac{1}{2}}\,dy = \frac{2}{3}y^{\frac{3}{2}}\ln y - \frac{4}{9}y^{\frac{3}{2}} + C$$

46) $\displaystyle\int \frac{\cos\theta}{1+\sin^2\theta}\,d\theta$

**Solution:** Using direct substitution with $u = \sin\theta$, and $du = \cos\theta\,d\theta$, we have that:

$$\int \frac{\cos\theta}{1+\sin^2\theta}\,d\theta = \int \frac{1}{1+u^2}\,du = \arctan u + C$$
$$\Rightarrow \int \frac{\cos\theta}{1+\sin^2\theta}\,d\theta = \arctan(\sin\theta) + C$$

47) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

**Solution:** Using inverse trigonometric substitution with $2\tan u = x$, and $2\sec u\,du = dx$, we get:

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{4\tan^2 u\sqrt{4\tan^2 u+4}}2\sec^2 u\,du = \int \frac{2\sec^2 u}{8\tan^2 u\sec u}\,du$$
$$= \int \frac{\cos^2 u}{4\cos u\sin^2 u}\,du = \int \frac{1}{4}\cot u\csc u\,du$$
$$= -\frac{1}{4}\csc u + C = -\frac{1}{4}\csc(\arctan\left(\frac{x}{2}\right)) + C$$
$$\Rightarrow \int \frac{1}{x^2\sqrt{x^2+4}}\,dx = -\frac{\sqrt{x^2+4}}{4x} + C$$

48) $\displaystyle\int te^{t^2}\,dt$

**Solution:** Using direct substitution with $u = t^2$, and $du = 2t\,dt$, we have that:

$$\int te^{t^2}\,dt = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u + C$$
$$\Rightarrow \int te^{t^2}\,dt = \frac{1}{2}e^{t^2} + C$$

49) $\displaystyle\int \cos(\pi t)\cos(\sin(\pi t))\,dt$

**Solution:** Using direct substitution with $u = \sin(\pi t)\,dt$, and $du = \pi\cos(\pi t)\,dt$, we have that:

$$\int \cos(\pi t)\cos(\sin(\pi t))\,dt = \int \frac{1}{\pi}\cos u\,du = \frac{1}{\pi}\sin u + C$$
$$\Rightarrow \int \cos(\pi t)\cos(\sin(\pi t))\,dt = \frac{1}{\pi}\sin(\cos(\pi t)) + C$$

50) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^5(x)\,dx$

**Solution:** Using direct substitution with $u = \cos x$, and $du = -\sin x\,dx$, when $x = 0$, then $u = 1$, and when $x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$. We have that:

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \sin^5(x)\,dx &= \int_0^{\frac{\pi}{4}} (\sin^2(x))^2\sin x\,dx = \int_0^{\frac{\pi}{4}} (1-\cos^2 x)^2\sin x\,dx \\
&= \int_1^{\frac{1}{\sqrt{2}}} -(1-u^2)^2\,du = \int_1^{\frac{1}{\sqrt{2}}} (-1+2u^2-u^4)\,du \\
&= (-u + \frac{2}{3}u^3 - \frac{1}{5}u^5)\,\Big|_1^{\frac{1}{\sqrt{2}}} \\
&= (-\frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{2}} - \frac{1}{20\sqrt{2}}) - (-1 + \frac{2}{3} - \frac{1}{5}) = -\frac{43}{60\sqrt{2}} + \frac{8}{15}
\end{aligned}$$
$$\Rightarrow \int_0^{\frac{\pi}{4}} \sin^5(x)\,dx = -\frac{43}{60\sqrt{2}} + \frac{8}{15}$$