Series

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

FOURIER SERIES

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A self-contained Tutorial Module for learning the technique of Fourier series analysis

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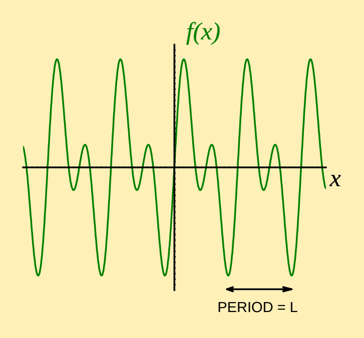
Table of contents

1. Theory

● A graph of **periodic** function $f(x)$ that has period L exhibits the same pattern every L units along the x-axis, so that $f(x + L) = f(x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods)



● This property of repetition defines a **fundamental spatial frequency** $k = \frac{2\pi}{L}$ that can be used to give a **first approximation** to the periodic pattern $f(x)$:

$$f(x) \simeq c_1 \sin(kx + \alpha_1) = a_1 \cos(kx) + b_1 \sin(kx),$$

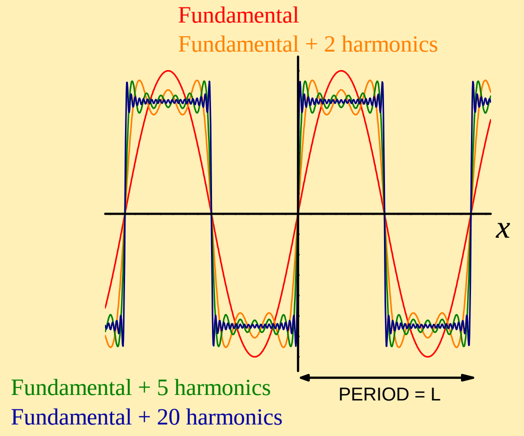
where symbols with subscript 1 are constants that determine the amplitude and phase of this first approximation

● A much **better approximation** of the periodic pattern $f(x)$ can be built up by adding an appropriate combination of **harmonics** to this fundamental (sine-wave) pattern. For example, adding

$$c_2 \sin(2kx + \alpha_2) = a_2 \cos(2kx) + b_2 \sin(2kx) \quad \text{(the 2nd harmonic)}$$
$$c_3 \sin(3kx + \alpha_3) = a_3 \cos(3kx) + b_3 \sin(3kx) \quad \text{(the 3rd harmonic)}$$

Here, symbols with subscripts are constants that determine the amplitude and phase of each harmonic contribution

One can even approximate a square-wave pattern with a suitable sum that involves a fundamental sine-wave plus a combination of harmonics of this fundamental frequency. This sum is called a **Fourier series**



● In this Tutorial, we consider working out Fourier series for functions $f(x)$ with period $L = 2\pi$. Their fundamental frequency is then $k = \frac{2\pi}{L} = 1$, and their Fourier series representations involve terms like

$$\begin{array}{ll} a_1 \cos x \;\;, & b_1 \sin x \\ a_2 \cos 2x \;, & b_2 \sin 2x \\ a_3 \cos 3x \;, & b_3 \sin 3x \end{array}$$

We also include a constant term $a_0/2$ in the Fourier series. This allows us to represent functions that are, for example, entirely above the $x-$axis. With a sufficient number of harmonics included, our approximate series can exactly represent a given function $f(x)$

$$\begin{aligned} f(x) = a_0/2 \quad &+ \quad a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + ... \\ &+ \quad b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + ... \end{aligned}$$

A more compact way of writing the Fourier series of a function $f(x)$, with period 2π, uses the variable subscript $n = 1, 2, 3, \ldots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

● We need to work out the **Fourier coefficients** (a_0, a_n and b_n) for given functions $f(x)$. This process is broken down into three steps

STEP ONE

$$a_0 = \frac{1}{\pi} \int_{2\pi} f(x)\, dx$$

STEP TWO

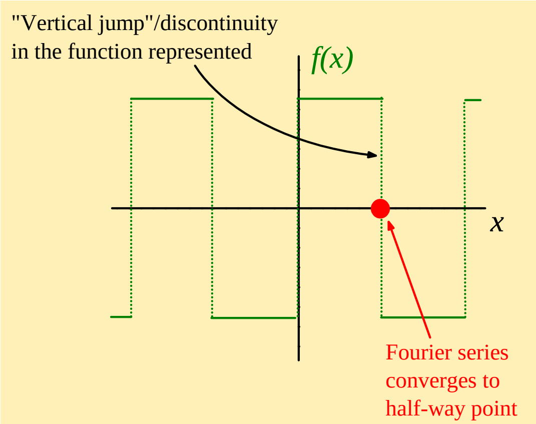
$$a_n = \frac{1}{\pi} \int_{2\pi} f(x) \cos nx\, dx$$

STEP THREE

$$b_n = \frac{1}{\pi} \int_{2\pi} f(x) \sin nx\, dx$$

where integrations are over a single interval in x of $L = 2\pi$

● Finally, specifying a particular value of $x = x_1$ in a Fourier series, gives a series of constants that should equal $f(x_1)$. However, if $f(x)$ is discontinuous at this value of x, then the series converges to a value that is **half-way** between the two possible function values



2. Exercises

Click on EXERCISE links for full worked solutions (7 exercises in total).

EXERCISE 1.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \ . \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$

b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right]$$

c) By giving an appropriate value to x, show that

$$\frac{\pi}{4} \quad = \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Exercise 2.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \;. \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$

b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right] + \left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right]$$

c) By giving appropriate values to x, show that

(i) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ and (ii) $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

Exercise 3.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi \end{cases}.$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$

b) Show that the Fourier series for $f(x)$ in the interval $0 < x < 2\pi$ is

$$\frac{3\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right]$$
$$- \left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots\right]$$

c) By giving appropriate values to x, show that

(i) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ and (ii) $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

Exercise 4.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \frac{x}{2} \text{ over the interval } 0 < x < 2\pi.$$

a) Sketch a graph of $f(x)$ in the interval $0 < x < 4\pi$

b) Show that the Fourier series for $f(x)$ in the interval $0 < x < 2\pi$ is

$$\frac{\pi}{2} - \left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots\right]$$

c) By giving an appropriate value to x, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

Exercise 5.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} \pi - x, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$

b) Show that the Fourier series for $f(x)$ in the interval $0 < x < 2\pi$ is

$$\begin{aligned} \frac{\pi}{4} \quad &+ \quad \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right] \\ &+ \quad \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \dots \end{aligned}$$

c) By giving an appropriate value to x, show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Exercise 6.

Let $f(x)$ be a function of period 2π such that

$$f(x) = x \text{ in the range } -\pi < x < \pi.$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$

b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\right]$$

c) By giving an appropriate value to x, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Exercise 7.

Let $f(x)$ be a function of period 2π such that

$$f(x) = x^2 \text{ over the interval } -\pi < x < \pi.$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$

b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\frac{\pi^2}{3} - 4\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots\right]$$

c) By giving an appropriate value to x, show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

3. Answers

The sketches asked for in part (a) of each exercise are given within the full worked solutions – click on the Exercise links to see these solutions

The answers below are suggested values of x to get the series of constants quoted in part (c) of each exercise

1. $x = \frac{\pi}{2}$,
2. (i) $x = \frac{\pi}{2}$, (ii) $x = 0$,
3. (i) $x = \frac{\pi}{2}$, (ii) $x = 0$,
4. $x = \frac{\pi}{2}$,
5. $x = 0$,
6. $x = \frac{\pi}{2}$,
7. $x = \pi$.

4. Integrals

Formula for integration by parts: $\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b \frac{du}{dx}v\,dx$

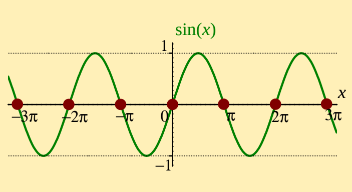
$f(x)$	$\int f(x)\mathrm{d}x$	$f(x)$	$\int f(x)\mathrm{d}x$
x^n	$\frac{x^{n+1}}{n+1}\quad(n\neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1}\quad(n\neq -1)$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{g'(x)}{g(x)}$	$\ln\lvert g(x)\rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a}\quad(a>0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\lvert\cos x\rvert$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\lvert\tan\frac{x}{2}\right\rvert$	$\operatorname{cosech} x$	$\ln\left\lvert\tanh\frac{x}{2}\right\rvert$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert$	$\operatorname{sech} x$	$2\tan^{-1}e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\lvert\sin x\rvert$	$\coth x$	$\ln\lvert\sinh x\rvert$
$\sin^2 x$	$\frac{x}{2}-\frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4}-\frac{x}{2}$
$\cos^2 x$	$\frac{x}{2}+\frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4}+\frac{x}{2}$

$f(x)$	$\int f(x)\,\mathrm{d}x$	$f(x)$	$\int f(x)\,\mathrm{d}x$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

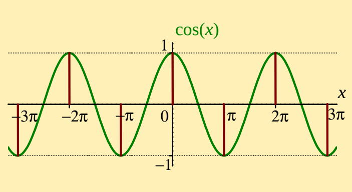
5. Useful trig results

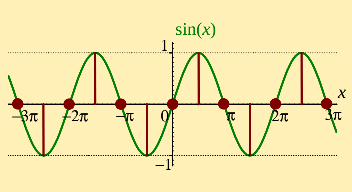
When calculating the Fourier coefficients a_n and b_n, for which $n = 1, 2, 3, \ldots$, the following trig. results are useful. Each of these results, which are also true for $n = 0, -1, -2, -3, \ldots$, can be deduced from the graph of $\sin x$ or that of $\cos x$

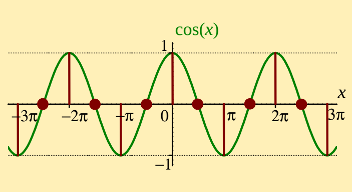
- $\sin n\pi = 0$



- $\cos n\pi = (-1)^n$



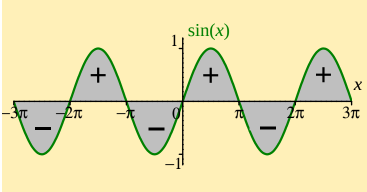




- $$\sin n\frac{\pi}{2} = \begin{cases} 0 & , n \text{ even} \\ 1 & , n = 1, 5, 9, \ldots \\ -1 & , n = 3, 7, 11, \ldots \end{cases}$$
- $$\cos n\frac{\pi}{2} = \begin{cases} 0 & , n \text{ odd} \\ 1 & , n = 0, 4, 8, \ldots \\ -1 & , n = 2, 6, 10, \ldots \end{cases}$$

Areas cancel when when integrating over whole periods

- $$\int_{2\pi} \sin nx \, \mathrm{d}x = 0$$
- $$\int_{2\pi} \cos nx \, \mathrm{d}x = 0$$



6. Alternative notation

● For a waveform $f(x)$ with period $L = \frac{2\pi}{k}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nkx + b_n \sin nkx]$$

The corresponding Fourier coefficients are

STEP ONE
$$a_0 = \frac{2}{L} \int_L f(x)\, dx$$

STEP TWO
$$a_n = \frac{2}{L} \int_L f(x) \cos nkx\, dx$$

STEP THREE
$$b_n = \frac{2}{L} \int_L f(x) \sin nkx\, dx$$

and integrations are over a single interval in x of L

● For a waveform $f(x)$ with period $2L = \frac{2\pi}{k}$, we have that $k = \frac{2\pi}{2L} = \frac{\pi}{L}$ and $nkx = \frac{n\pi x}{L}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

The corresponding Fourier coefficients are

STEP ONE

$$a_0 = \frac{1}{L}\int_{2L} f(x)\,dx$$

STEP TWO

$$a_n = \frac{1}{L}\int_{2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

STEP THREE

$$b_n = \frac{1}{L}\int_{2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

and integrations are over a single interval in x of $2L$

● For a waveform $f(t)$ with period $T = \frac{2\pi}{\omega}$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\omega t + b_n \sin n\omega t]$$

The corresponding Fourier coefficients are

STEP ONE
$$a_0 = \frac{2}{T} \int_T f(t)\, dt$$

STEP TWO
$$a_n = \frac{2}{T} \int_T f(t) \cos n\omega t\, dt$$

STEP THREE
$$b_n = \frac{2}{T} \int_T f(t) \sin n\omega t\, dt$$

and integrations are over a single interval in t of T

7. Tips on using solutions

- When looking at the THEORY, ANSWERS, INTEGRALS, TRIG or NOTATION pages, use the Back button (at the bottom of the page) to return to the exercises

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

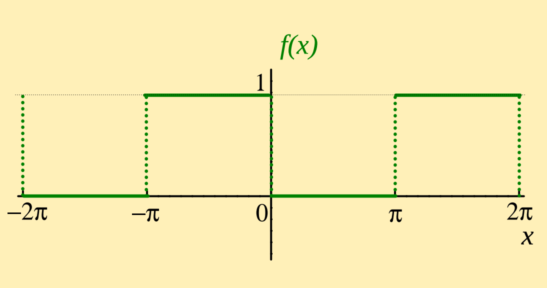
- Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1.

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi, \end{cases} \quad \text{and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$
\begin{aligned}
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \mathrm{d}x \\
&= \frac{1}{\pi}\left[x\right]_{-\pi}^{0} \\
&= \frac{1}{\pi}\left(0-(-\pi)\right) \\
&= \frac{1}{\pi}\cdot(\pi) \\
\text{i.e. } a_0 &= 1\,.
\end{aligned}
$$

STEP TWO

$$
\begin{aligned}
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} = \frac{1}{n\pi}\left[\sin nx\right]_{-\pi}^{0} \\
&= \frac{1}{n\pi}\left(\sin 0 - \sin(-n\pi)\right) \\
&= \frac{1}{n\pi}(0 + \sin n\pi) \\
\text{i.e. } a_n &= \frac{1}{n\pi}(0 + 0) = 0.
\end{aligned}
$$

STEP THREE

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\sin nx\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} \text{i.e.}\quad b_n &= \frac{1}{\pi}\int_{-\pi}^{0}\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} \\ &= -\frac{1}{n\pi}[\cos nx]_{-\pi}^{0} = -\frac{1}{n\pi}\left(\cos 0 - \cos(-n\pi)\right) \\ &= -\frac{1}{n\pi}(1-\cos n\pi) = -\frac{1}{n\pi}\left(1-(-1)^n\right),\ \text{see Trig} \\ \text{i.e.}\quad b_n &= \begin{cases} 0 & ,\ n \text{ even} \\ -\frac{2}{n\pi} & ,\ n \text{ odd} \end{cases} \quad ,\ \text{since } (-1)^n = \begin{cases} 1 & ,\ n \text{ even} \\ -1 & ,\ n \text{ odd} \end{cases} \end{aligned}$$

We now have that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

with the three steps giving

$$a_0 = 1, \; a_n = 0 \, , \text{ and } \; b_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n\pi} & , n \text{ odd} \end{cases}$$

It may be helpful to construct a table of values of b_n

n	1	2	3	4	5
b_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi}\left(\frac{1}{3}\right)$	0	$-\frac{2}{\pi}\left(\frac{1}{5}\right)$

Substituting our results now gives the required series

$$f(x) = \frac{1}{2} - \frac{2}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \ldots \right]$$

c) Pick an appropriate value of x, to show that

$$\boxed{\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \dots}$$

Comparing this series with

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right],$$

we need to introduce a minus sign in front of the constants $\frac{1}{3}$, $\frac{1}{7}$,...

So we need $\sin x = 1$, $\sin 3x = -1$, $\sin 5x = 1$, $\sin 7x = -1$, etc

The first condition of $\sin x = 1$ suggests trying $x = \frac{\pi}{2}$.

This choice gives	$\sin\frac{\pi}{2}$	$+$	$\frac{1}{3}\sin 3\frac{\pi}{2}$	$+$	$\frac{1}{5}\sin 5\frac{\pi}{2}$	$+$	$\frac{1}{7}\sin 7\frac{\pi}{2}$
i.e.	1	$-$	$\frac{1}{3}$	$+$	$\frac{1}{5}$	$-$	$\frac{1}{7}$

Looking at the graph of $f(x)$, we also have that $f(\frac{\pi}{2}) = 0$.

Picking $x = \frac{\pi}{2}$ thus gives

$$\begin{aligned} 0 &= \tfrac{1}{2} - \tfrac{2}{\pi}\Big[\sin\tfrac{\pi}{2} + \tfrac{1}{3}\sin\tfrac{3\pi}{2} + \tfrac{1}{5}\sin\tfrac{5\pi}{2} \\ &\qquad + \tfrac{1}{7}\sin\tfrac{7\pi}{2} + \ldots\Big] \\ \text{i.e. } 0 &= \tfrac{1}{2} - \tfrac{2}{\pi}\Big[1 - \tfrac{1}{3} + \tfrac{1}{5} \\ &\qquad - \tfrac{1}{7} + \ldots\Big] \end{aligned}$$

A little manipulation then gives a series representation of $\frac{\pi}{4}$

$$\frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right] = \frac{1}{2}$$

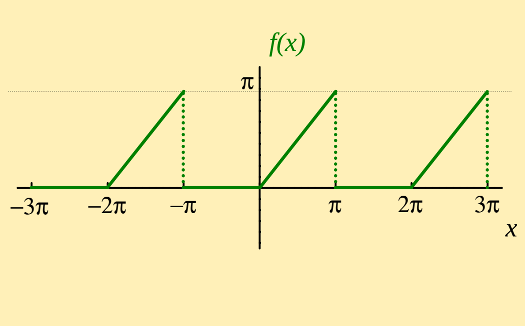
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.$$

Return to Exercise 1

Exercise 2.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi, \end{cases} \quad \text{and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$
\begin{aligned}
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot \mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) \\
\text{i.e. } a_0 &= \frac{\pi}{2}\,.
\end{aligned}
$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x$$

i.e. $$a_n = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x = \frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,\mathrm{d}x\right\}$$

(using integration by parts)

i.e. $$a_n = \frac{1}{\pi}\left\{\left(\pi\frac{\sin n\pi}{n} - 0\right) - \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{(\quad 0 \quad - \ 0) + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi n^2}\left\{\cos n\pi - \cos 0\right\} = \frac{1}{\pi n^2}\left\{(-1)^n - 1\right\}$$

i.e. $$a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{\pi n^2} & , n \text{ odd} \end{cases}$$, see Trig.

STEP THREE

$$\begin{aligned}
b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x \\
\text{i.e. } b_n = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x &= \frac{1}{\pi}\left\{\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos nx}{n}\right)\mathrm{d}x\right\}
\end{aligned}$$

(using integration by parts)

$$\begin{aligned}
&= \frac{1}{\pi}\left\{-\frac{1}{n}\left[x\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,\mathrm{d}x\right\} \\
&= \frac{1}{\pi}\left\{-\frac{1}{n}\left(\pi\cos n\pi - 0\right) + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\} \\
&= -\frac{1}{n}(-1)^n + \frac{1}{\pi n^2}(0-0), \text{ see TRIG} \\
&= -\frac{1}{n}(-1)^n
\end{aligned}$$

i.e. $$b_n = \begin{cases} -\frac{1}{n} & , n \text{ even} \\ +\frac{1}{n} & , n \text{ odd} \end{cases}$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $a_0 = \frac{\pi}{2}$, $a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{\pi n^2} & , n \text{ odd} \end{cases}$, $b_n = \begin{cases} -\frac{1}{n} & , n \text{ even} \\ \frac{1}{n} & , n \text{ odd} \end{cases}$

Constructing a table of values gives

n	1	2	3	4	5
a_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi} \cdot \frac{1}{3^2}$	0	$-\frac{2}{\pi} \cdot \frac{1}{5^2}$
b_n	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$

This table of coefficients gives

$$\begin{aligned} f(x) = \frac{1}{2}\left(\frac{\pi}{2}\right) \quad &+ \left(-\frac{2}{\pi}\right)\cos x + 0\cdot\cos 2x \\ &+ \left(-\frac{2}{\pi}\cdot\frac{1}{3^2}\right)\cos 3x + 0\cdot\cos 4x \\ &+ \left(-\frac{2}{\pi}\cdot\frac{1}{5^2}\right)\cos 5x + ... \\ &+ \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - ... \end{aligned}$$

$$\begin{aligned} \text{i.e. } f(x) = \frac{\pi}{4} \quad &- \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...\right] \\ &+ \left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - ...\right] \end{aligned}$$

and we have found the required series!

c) Pick an appropriate value of x, to show that

$$\boxed{\text{(i)}\ \tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots}$$

Comparing this series with

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right] + \left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right],$$

the required series of constants does not involve terms like $\frac{1}{3^2}, \frac{1}{5^2}, \frac{1}{7^2}, \ldots$. So we need to pick a value of x that sets the $\cos nx$ terms to zero. The TRIG section shows that $\cos n\frac{\pi}{2} = 0$ when n is odd, and note also that $\cos nx$ terms in the Fourier series all have odd n

i.e. $\cos x = \cos 3x = \cos 5x = \ldots = 0$ when $x = \frac{\pi}{2}$,

i.e. $\cos\frac{\pi}{2} = \cos 3\frac{\pi}{2} = \cos 5\frac{\pi}{2} = \ldots = 0$

Setting $x = \frac{\pi}{2}$ in the series for $f(x)$ gives

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \frac{\pi}{4} - \frac{2}{\pi}\left[\cos\frac{\pi}{2} + \frac{1}{3^2}\cos\frac{3\pi}{2} + \frac{1}{5^2}\cos\frac{5\pi}{2} + \ldots\right] \\ &\quad + \left[\sin\frac{\pi}{2} - \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} - \frac{1}{4}\sin\frac{4\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} - \ldots\right] \\ &= \frac{\pi}{4} - \frac{2}{\pi}[0 \; + \; 0 \; + \; 0 \; + \ldots] \\ &\quad + \left[1 - \frac{1}{2}\underbrace{\sin\pi}_{=0} + \frac{1}{3}\cdot(-1) - \frac{1}{4}\underbrace{\sin 2\pi}_{=0} + \frac{1}{5}\cdot(1) - \ldots\right] \end{aligned}$$

The graph of $f(x)$ shows that $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, so that

$$\begin{aligned} \frac{\pi}{2} &= \frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \\ \text{i.e. } \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \end{aligned}$$

Pick an appropriate value of x, to show that

$$\boxed{\text{(ii)}\ \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots}$$

Compare this series with

$$f(x) = \frac{\pi}{4} \quad - \quad \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$$
$$+ \quad \left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right].$$

This time, we want to use the coefficients of the $\cos nx$ terms, and the same choice of x needs to set the $\sin nx$ terms to zero

Picking $x = 0$ gives

$$\sin x = \sin 2x = \sin 3x = 0 \quad \text{and} \quad \cos x = \cos 3x = \cos 5x = 1$$

Note also that the graph of $f(x)$ gives $f(x) = 0$ when $x = 0$

So, picking $x = 0$ gives

$$\begin{aligned} 0 &= \frac{\pi}{4} - \frac{2}{\pi}\left[\cos 0 + \frac{1}{3^2}\cos 0 + \frac{1}{5^2}\cos 0 + \frac{1}{7^2}\cos 0 + ...\right] \\ &\quad + \sin 0 - \frac{\sin 0}{2} + \frac{\sin 0}{3} - ... \\ \text{i.e. } 0 &= \frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...\right] + 0 - 0 + 0 - ... \end{aligned}$$

We then find that

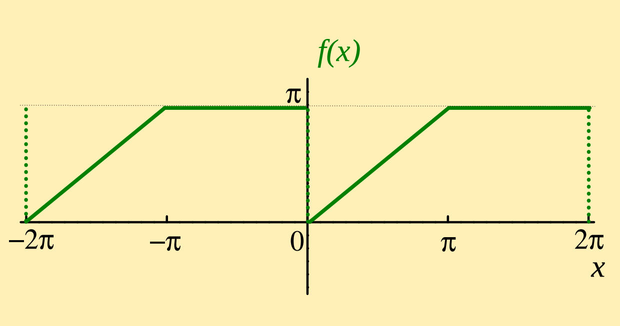
$$\begin{aligned} \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...\right] &= \frac{\pi}{4} \\ \text{and} \qquad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... &= \frac{\pi^2}{8}. \end{aligned}$$

Return to Exercise 2

Exercise 3.

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi, \end{cases} \quad \text{and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$
\begin{aligned}
a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x + \frac{1}{\pi}\int_\pi^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x\mathrm{d}x + \frac{1}{\pi}\int_\pi^{2\pi} \pi\cdot\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{\pi}{\pi}\Big[x\Big]_\pi^{2\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) + \Big(2\pi - \pi\Big) \\
&= \frac{\pi}{2} + \pi \\
\text{i.e. } a_0 &= \frac{3\pi}{2}.
\end{aligned}
$$

STEP TWO

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\cos nx\,\mathrm{d}x \\
&= \underbrace{\frac{1}{\pi}\left[\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\mathrm{d}x\right]}_{\text{using integration by parts}} + \frac{\pi}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi} \\
&= \frac{1}{\pi}\left[\frac{1}{n}\Big(\pi\sin n\pi - 0\cdot\sin n0\Big) - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi}\right] \\
&\qquad + \frac{1}{n}(\sin n2\pi - \sin n\pi)
\end{aligned}
$$

$$\text{i.e. } a_n = \frac{1}{\pi}\left[\frac{1}{n}\Big(0-0\Big) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] + \frac{1}{n}\Big(0-0\Big)$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1), \quad \text{see TRIG}$$

$$= \frac{1}{n^2\pi}\big((-1)^n - 1\big),$$

$$\text{i.e. } a_n = \begin{cases} -\frac{2}{n^2\pi} & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases}$$

STEP THREE

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\sin nx\,\mathrm{d}x \\
&= \underbrace{\frac{1}{\pi}\left[\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos nx}{n}\right)\mathrm{d}x\right]}_{\text{using integration by parts}} + \frac{\pi}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi} \\
&= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}+0\right)+\left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right] - \frac{1}{n}(\cos 2n\pi - \cos n\pi) \\
&= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left(\frac{\sin n\pi - \sin 0}{n^2}\right)\right] - \frac{1}{n}\left(1-(-1)^n\right) \\
&= -\frac{1}{n}(-1)^n + 0 - \frac{1}{n}\left(1-(-1)^n\right)
\end{aligned}$$

$$\text{i.e. } b_n = -\frac{1}{n}(-1)^n - \frac{1}{n} + \frac{1}{n}(-1)^n$$
$$\text{i.e. } b_n = -\frac{1}{n}.$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $a_0 = \frac{3\pi}{2}$, $a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n^2\pi} & , n \text{ odd} \end{cases}$, $b_n = -\frac{1}{n}$

Constructing a table of values gives

n	1	2	3	4	5
a_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi}\left(\frac{1}{3^2}\right)$	0	$-\frac{2}{\pi}\left(\frac{1}{5^2}\right)$
b_n	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{5}$

This table of coefficients gives

$$f(x) = \tfrac{1}{2}\left(\tfrac{3\pi}{2}\right) + \left(-\tfrac{2}{\pi}\right)\left[\cos x + 0\cdot\cos 2x + \tfrac{1}{3^2}\cos 3x + \dots\right]$$

$$+ \left(-1\right)\left[\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \dots\right]$$

$$\text{i.e. } f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos x + \tfrac{1}{3^2}\cos 3x + \tfrac{1}{5^2}\cos 5x + \dots\right]$$

$$- \left[\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \dots\right]$$

and we have found the required series.

c) Pick an appropriate value of x, to show that

$$\boxed{\text{(i) } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots}$$

Compare this series with

$$f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$$
$$- \left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right]$$

Here, we want to set the $\cos nx$ terms to zero (since their coefficients are $1, \frac{1}{3^2}, \frac{1}{5^2}, \ldots$). Since $\cos n\frac{\pi}{2} = 0$ when n is odd, we will try setting $x = \frac{\pi}{2}$ in the series. Note also that $f(\frac{\pi}{2}) = \frac{\pi}{2}$

This gives

$$\frac{\pi}{2} = \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos\frac{\pi}{2} + \frac{1}{3^2}\cos 3\frac{\pi}{2} + \frac{1}{5^2}\cos 5\frac{\pi}{2} + \ldots\right]$$
$$- \left[\sin\frac{\pi}{2} + \frac{1}{2}\sin 2\frac{\pi}{2} + \frac{1}{3}\sin 3\frac{\pi}{2} + \frac{1}{4}\sin 4\frac{\pi}{2} + \frac{1}{5}\sin 5\frac{\pi}{2} + \ldots\right]$$

and

$$\frac{\pi}{2} = \frac{3\pi}{4} - \frac{2}{\pi}\left[0 + 0 + 0 + \ldots\right]$$

$$- \quad \left[(1) + \tfrac{1}{2}\cdot(0) + \tfrac{1}{3}\cdot(-1) + \tfrac{1}{4}\cdot(0) + \tfrac{1}{5}\cdot(1) + \ldots\right]$$

then

$$\frac{\pi}{2} = \frac{3\pi}{4} - \left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots\right)$$

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots = \frac{3\pi}{4} - \frac{\pi}{2}$$

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots = \frac{\pi}{4}, \qquad \text{as required.}$$

To show that

$$\boxed{\text{(ii) } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots},$$

We want zero $\sin nx$ terms and to use the coefficients of $\cos nx$

Setting $x = 0$ eliminates the $\sin nx$ terms from the series, and also gives

$$\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \frac{1}{7^2}\cos 7x + \ldots = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

(i.e. the desired series).

The graph of $f(x)$ shows a discontinuity (a "vertical jump") at $x = 0$

The Fourier series converges to a value that is **half-way** between the two values of $f(x)$ around this discontinuity. That is the series will converge to $\frac{\pi}{2}$ at $x = 0$

$$\begin{aligned} \text{i.e. } \frac{\pi}{2} &= \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos 0 + \frac{1}{3^2}\cos 0 + \frac{1}{5^2}\cos 0 + \frac{1}{7^2}\cos 0 + \ldots\right] \\ &\quad - \left[\sin 0 + \frac{1}{2}\sin 0 + \frac{1}{3}\sin 0 + \ldots\right] \\ \text{and } \frac{\pi}{2} &= \frac{3\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right] - [0 + 0 + 0 + \ldots] \end{aligned}$$

Finally, this gives

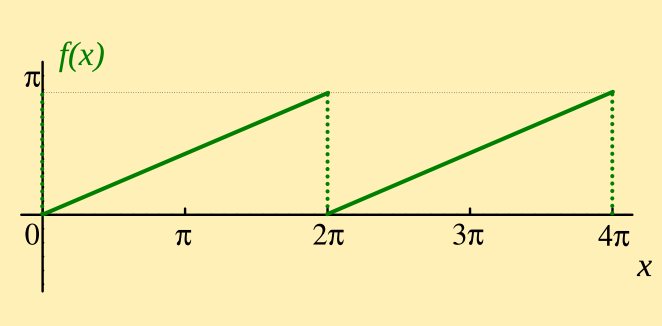
$$
\begin{aligned}
-\frac{\pi}{4} &= -\frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right) \\
\text{and} \quad \frac{\pi^2}{8} &= 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots
\end{aligned}
$$

Return to Exercise 3

Exercise 4.

$f(x) = \frac{x}{2}$, over the interval $0 < x < 2\pi$ and has period 2π

a) Sketch a graph of $f(x)$ in the interval $0 < x < 4\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2}\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{x^2}{4}\right]_0^{2\pi} \\ &= \frac{1}{\pi}\left[\frac{(2\pi)^2}{4} - 0\right] \\ \text{i.e. } a_0 &= \pi. \end{aligned}$$

STEP TWO

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2}\cos nx\,\mathrm{d}x \\
&= \underbrace{\frac{1}{2\pi}\left\{\left[x\frac{\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nx\,\mathrm{d}x\right\}}_{\text{using integration by parts}} \\
&= \frac{1}{2\pi}\left\{\left(2\pi\frac{\sin n2\pi}{n} - 0\cdot\frac{\sin n\cdot 0}{n}\right) - \frac{1}{n}\cdot 0\right\} \\
&= \frac{1}{2\pi}\left\{(0-0) - \frac{1}{n}\cdot 0\right\} \quad \text{, see Trig} \\
\text{i.e. } a_n &= 0.
\end{aligned}
$$

STEP THREE

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)\sin nx\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} x\sin nx\,\mathrm{d}x \\
&= \frac{1}{2\pi}\underbrace{\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{2\pi} - \int_0^{2\pi}\left(\frac{-\cos nx}{n}\right)\mathrm{d}x\right\}}_{\text{using integration by parts}} \\
&= \frac{1}{2\pi}\left\{\frac{1}{n}\left(-2\pi\cos n2\pi + 0\right) + \frac{1}{n}\cdot 0\right\}, \text{ see TRIG} \\
&= \frac{-2\pi}{2\pi n}\cos(n2\pi) \\
&= -\frac{1}{n}\cos(2n\pi) \\
\text{i.e. } b_n &= -\frac{1}{n} \quad \text{, since } 2n \text{ is even (see TRIG)}
\end{aligned}
$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $a_0 = \pi,\ a_n = 0,\ b_n = -\frac{1}{n}$

These Fourier coefficients give

$$\begin{aligned} f(x) &= \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(0 - \frac{1}{n} \sin nx\right) \\ \text{i.e. } f(x) &= \frac{\pi}{2} - \left\{ \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \ldots \right\}. \end{aligned}$$

c) Pick an appropriate value of x, to show that

$$\boxed{\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \ldots}$$

Setting $x = \frac{\pi}{2}$ gives $f(x) = \frac{\pi}{4}$ and

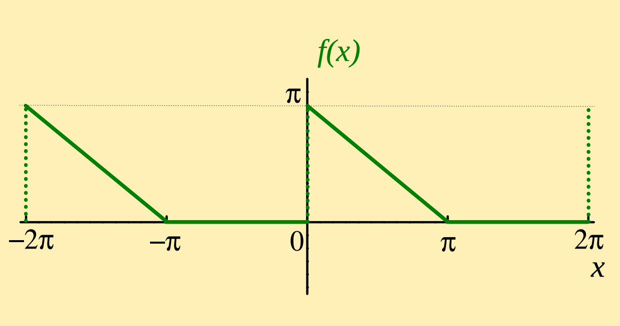
$$\begin{aligned} \frac{\pi}{4} &= \frac{\pi}{2} - \left[1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \ldots\right] \\ \frac{\pi}{4} &= \frac{\pi}{2} - \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right] \\ \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right] &= \frac{\pi}{4} \\ \text{i.e.} \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots &= \frac{\pi}{4}. \end{aligned}$$

Return to Exercise 4

Exercise 5.

$$f(x) = \begin{cases} \pi - x & , 0 < x < \pi \\ 0 & , \pi < x < 2\pi, \text{ and has period } 2\pi \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} 0\cdot \mathrm{d}x \\ &= \frac{1}{\pi}\left[\pi x - \frac{1}{2}x^2\right]_0^{\pi} + 0 \\ &= \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2} - 0\right] \\ \text{i.e. } a_0 &= \frac{\pi}{2}. \end{aligned}$$

STEP TWO

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} 0\cdot\mathrm{d}x \\
\text{i.e. } a_n &= \frac{1}{\pi}\underbrace{\left\{\left[(\pi - x)\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}(-1)\cdot\frac{\sin nx}{n}\,\mathrm{d}x\right\}}_{\text{using integration by parts}} + 0 \\
&= \frac{1}{\pi}\left\{(0-0) + \int_0^{\pi}\frac{\sin nx}{n}\mathrm{d}x\right\} \quad \text{, see Trig} \\
&= \frac{1}{\pi n}\left[\frac{-\cos nx}{n}\right]_0^{\pi} \\
&= -\frac{1}{\pi n^2}(\cos n\pi - \cos 0) \\
\text{i.e. } a_n &= -\frac{1}{\pi n^2}\left((-1)^n - 1\right) \quad \text{, see Trig}
\end{aligned}$$

$$\text{i.e. } a_n = \begin{cases} 0 & , n \text{ even} \\ \frac{2}{\pi n^2} & , n \text{ odd} \end{cases}$$

STEP THREE

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 0\cdot\mathrm{d}x \\ &= \frac{1}{\pi}\left\{\left[(\pi - x)\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}(-1)\cdot\left(-\frac{\cos nx}{n}\right)\mathrm{d}x\right\} + 0 \\ &= \frac{1}{\pi}\left\{\left(0 - \left(-\frac{\pi}{n}\right)\right) - \frac{1}{n}\cdot 0\right\}, \text{ see Trig} \\ \text{i.e. } b_n &= \frac{1}{n}. \end{aligned}$$

In summary, $a_0 = \frac{\pi}{2}$ and a table of other Fourier cofficients is

n	1	2	3	4	5
$a_n = \frac{2}{\pi n^2}$ (when n is odd)	$\frac{2}{\pi}$	0	$\frac{2}{\pi}\frac{1}{3^2}$	0	$\frac{2}{\pi}\frac{1}{5^2}$
$b_n = \frac{1}{n}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

$$\begin{aligned}
\therefore\ f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx] \\
&= \frac{\pi}{4} + \frac{2}{\pi}\cos x + \frac{2}{\pi}\frac{1}{3^2}\cos 3x + \frac{2}{\pi}\frac{1}{5^2}\cos 5x + \ldots \\
&\quad + \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \ldots \\
\text{i.e. } f(x) &= \frac{\pi}{4} + \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right] \\
&\quad + \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \ldots
\end{aligned}$$

c) To show that $\boxed{\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots}$,

note that, as $x \to 0$, the series converges to the half-way value of $\frac{\pi}{2}$,

$$\text{and then} \quad \frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left(\cos 0 + \frac{1}{3^2}\cos 0 + \frac{1}{5^2}\cos 0 + \dots\right)$$
$$+ \sin 0 + \frac{1}{2}\sin 0 + \frac{1}{3}\sin 0 + \dots$$

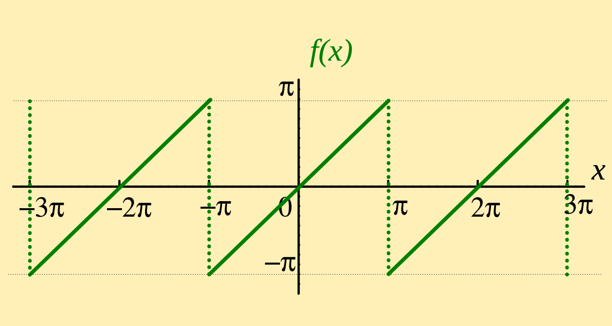
$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) + 0$$
$$\frac{\pi}{4} = \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$
$$\text{giving} \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Return to Exercise 5

Exercise 6.

$$\boxed{f(x) = x, \text{ over the interval } -\pi < x < \pi \text{ and has period } 2\pi}$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left(\frac{\pi^2}{2}-\frac{\pi^2}{2}\right) \\ \text{i.e. } a_0 &= 0. \end{aligned}$$

STEP TWO

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,\mathrm{d}x \\
&= \underbrace{\frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{\sin nx}{n}\right)\mathrm{d}x\right\}}_{\text{using integration by parts}} \\
\text{i.e. } a_n &= \frac{1}{\pi}\left\{\frac{1}{n}\left(\pi\sin n\pi - (-\pi)\sin(-n\pi)\right) - \frac{1}{n}\int_{-\pi}^{\pi}\sin nx\,\mathrm{d}x\right\} \\
&= \frac{1}{\pi}\left\{\frac{1}{n}(0-0) - \frac{1}{n}\cdot 0\right\}, \\
&\qquad \text{since } \sin n\pi = 0 \text{ and } \int_{2\pi}\sin nx\,\mathrm{d}x = 0, \\
\text{i.e. } a_n &= 0.
\end{aligned}$$

STEP THREE

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left\{\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{-\cos nx}{n}\right)\,\mathrm{d}x\right\} \\
&= \frac{1}{\pi}\left\{-\frac{1}{n}\left[x\cos nx\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx\,\mathrm{d}x\right\} \\
&= \frac{1}{\pi}\left\{-\frac{1}{n}\left(\pi\cos n\pi - (-\pi)\cos(-n\pi)\right) + \frac{1}{n}\cdot 0\right\} \\
&= -\frac{\pi}{n\pi}\left(\cos n\pi + \cos n\pi\right) \\
&= -\frac{1}{n}\left(2\cos n\pi\right) \\
\text{i.e. } b_n &= -\frac{2}{n}(-1)^n.
\end{aligned}$$

We thus have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

with $a_0 = 0,\ a_n = 0,\ b_n = -\frac{2}{n}(-1)^n$

and

n	1	2	3
b_n	2	-1	$\frac{2}{3}$

Therefore

$$\begin{aligned} f(x) &= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots \\ \text{i.e. } f(x) &= 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right] \end{aligned}$$

and we have found the required Fourier series.

c) Pick an appropriate value of x, to show that

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots}$$

Setting $x = \frac{\pi}{2}$ gives $f(x) = \frac{\pi}{2}$ and

$$\frac{\pi}{2} = 2\left[\sin\frac{\pi}{2} - \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} - \frac{1}{4}\sin\frac{4\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} - \ldots\right]$$

This gives

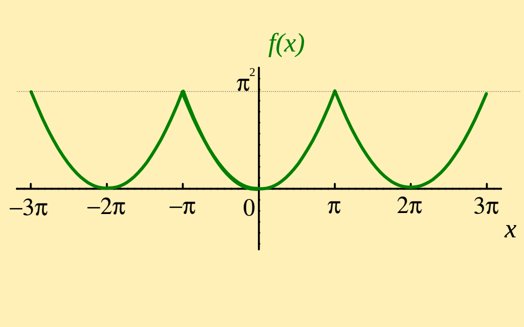
$$\begin{aligned} \frac{\pi}{2} &= 2\left[1 + 0 + \frac{1}{3}\cdot(-1) - 0 + \frac{1}{5}\cdot(1) - 0 + \frac{1}{7}\cdot(-1) + \ldots\right] \\ \frac{\pi}{2} &= 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right] \\ \text{i.e.} \quad \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \end{aligned}$$

Return to Exercise 6

Exercise 7.

$f(x) = x^2$, over the interval $-\pi < x < \pi$ and has period 2π

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned}
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^3}{3}-\left(-\frac{\pi^3}{3}\right)\right) \\
&= \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) \\
\text{i.e. } a_0 &= \frac{2\pi^2}{3}.
\end{aligned}$$

STEP TWO

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,\mathrm{d}x \\
&= \underbrace{\frac{1}{\pi}\left\{\left[x^2\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\left(\frac{\sin nx}{n}\right)\mathrm{d}x\right\}}_{\text{using integration by parts}} \\
&= \frac{1}{\pi}\left\{\frac{1}{n}\left(\pi^2\sin n\pi - \pi^2\sin(-n\pi)\right) - \frac{2}{n}\int_{-\pi}^{\pi} x\sin nx\,\mathrm{d}x\right\} \\
&= \frac{1}{\pi}\left\{\frac{1}{n}(0-0) - \frac{2}{n}\int_{-\pi}^{\pi} x\sin nx\,\mathrm{d}x\right\},\ \text{see TRIG} \\
&= \frac{-2}{n\pi}\int_{-\pi}^{\pi} x\sin nx\,\mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
\text{i.e. } a_n &= \frac{-2}{n\pi}\underbrace{\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{-\cos nx}{n}\right)\mathrm{d}x\right\}}_{\text{using integration by parts again}} \\
&= \frac{-2}{n\pi}\left\{-\frac{1}{n}\left[x\cos nx\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx\,\mathrm{d}x\right\} \\
&= \frac{-2}{n\pi}\left\{-\frac{1}{n}\left(\pi\cos n\pi - (-\pi)\cos(-n\pi)\right) + \frac{1}{n}\cdot 0\right\} \\
&= \frac{-2}{n\pi}\left\{-\frac{1}{n}\left(\pi(-1)^n + \pi(-1)^n\right)\right\} \\
&= \frac{-2}{n\pi}\left\{\frac{-2\pi}{n}(-1)^n\right\}
\end{aligned}$$

$$\text{i.e. } a_n = \frac{-2}{n\pi}\left\{-\frac{2\pi}{n}(-1)^n\right\}$$
$$= \frac{+4\pi}{\pi n^2}(-1)^n$$
$$= \frac{4}{n^2}(-1)^n$$

$$\text{i.e. } a_n = \begin{cases} \frac{4}{n^2} & , n \text{ even} \\ \frac{-4}{n^2} & , n \text{ odd.} \end{cases}$$

STEP THREE

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\underbrace{\left\{\left[x^2\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\cdot\left(\frac{-\cos nx}{n}\right)\mathrm{d}x\right\}}_{\text{using integration by parts}} \\
&= \frac{1}{\pi}\left\{-\frac{1}{n}\left[x^2\cos nx\right]_{-\pi}^{\pi} + \frac{2}{n}\int_{-\pi}^{\pi} x\cos nx\,\mathrm{d}x\right\} \\
&= \frac{1}{\pi}\left\{-\frac{1}{n}\left(\pi^2\cos n\pi - \pi^2\cos(-n\pi)\right) + \frac{2}{n}\int_{-\pi}^{\pi} x\cos nx\,\mathrm{d}x\right\} \\
&= \frac{1}{\pi}\left\{-\frac{1}{n}\underbrace{\left(\pi^2\cos n\pi - \pi^2\cos(n\pi)\right)}_{=0} + \frac{2}{n}\int_{-\pi}^{\pi} x\cos nx\,\mathrm{d}x\right\} \\
&= \frac{2}{\pi n}\int_{-\pi}^{\pi} x\cos nx\,\mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
\text{i.e. } b_n &= \frac{2}{\pi n} \underbrace{\left\{ \left[x \frac{\sin nx}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin nx}{n} \, \mathrm{d}x \right\}}_{\text{using integration by parts}} \\
&= \frac{2}{\pi n} \left\{ \frac{1}{n} \left(\pi \sin n\pi - (-\pi) \sin(-n\pi) \right) - \frac{1}{n} \int_{-\pi}^{\pi} \sin nx \, \mathrm{d}x \right\} \\
&= \frac{2}{\pi n} \left\{ \frac{1}{n} (0 + 0) - \frac{1}{n} \int_{-\pi}^{\pi} \sin nx \, \mathrm{d}x \right\} \\
&= \frac{-2}{\pi n^2} \int_{-\pi}^{\pi} \sin nx \, \mathrm{d}x \\
\text{i.e. } b_n &= 0.
\end{aligned}$$

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $a_0 = \frac{2\pi^2}{3}$, $a_n = \begin{cases} \frac{4}{n^2} & , n \text{ even} \\ \frac{-4}{n^2} & , n \text{ odd} \end{cases}$, $b_n = 0$

n	1	2	3	4
a_n	$-4(1)$	$4\left(\frac{1}{2^2}\right)$	$-4\left(\frac{1}{3^2}\right)$	$4\left(\frac{1}{4^2}\right)$

$$\text{i.e. } f(x) = \frac{1}{2}\left(\frac{2\pi^2}{3}\right) - 4\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x \ldots\right] + [0 + 0 + 0 + \ldots]$$

$$\text{i.e. } f(x) = \frac{\pi^2}{3} - 4\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x + \ldots\right].$$

c) To show that $\boxed{\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots}$,

use the fact that $\cos n\pi = \begin{cases} 1 & , n \text{ even} \\ -1 & , n \text{ odd} \end{cases}$

$$\begin{array}{llllllllll} \text{i.e.} & \cos x & - & \frac{1}{2^2}\cos 2x & + & \frac{1}{3^2}\cos 3x & - & \frac{1}{4^2}\cos 4x & +\ldots & \text{with } x=\pi \\ \text{gives} & \cos \pi & - & \frac{1}{2^2}\cos 2\pi & + & \frac{1}{3^2}\cos 3\pi & - & \frac{1}{4^2}\cos 4\pi & +\ldots & \\ \text{i.e.} & (-1) & - & \frac{1}{2^2}\cdot(1) & + & \frac{1}{3^2}\cdot(-1) & - & \frac{1}{4^2}\cdot(1) & +\ldots & \\ \text{i.e.} & -1 & - & \frac{1}{2^2} & - & \frac{1}{3^2} & - & \frac{1}{4^2} & +\ldots & \end{array}$$

$$= -1 \cdot \underbrace{\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right)}_{\text{(the desired series)}}$$

The graph of $f(x)$ gives that $f(\pi) = \pi^2$ and the series converges to this value.

Setting $x = \pi$ in the Fourier series thus gives

$$\begin{aligned}
\pi^2 &= \frac{\pi^2}{3} - 4\left(\cos\pi - \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi - \frac{1}{4^2}\cos 4\pi + \ldots\right) \\
\pi^2 &= \frac{\pi^2}{3} - 4\left(-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \ldots\right) \\
\pi^2 &= \frac{\pi^2}{3} + 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) \\
\frac{2\pi^2}{3} &= 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) \\
\text{i.e. } \frac{\pi^2}{6} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots
\end{aligned}$$

Return to Exercise 7

Lecture 11
The Fourier transform

- definition
- examples
- the Fourier transform of a unit step
- the Fourier transform of a periodic signal
- properties
- the inverse Fourier transform

The Fourier transform

we'll be interested in signals defined for all t

the **Fourier transform** of a signal f is the function

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

- F is a function of a *real* variable ω; the function value $F(\omega)$ is (in general) a complex number

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\,dt - j\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

- $|F(\omega)|$ is called the *amplitude spectrum* of f; $\angle F(\omega)$ is the *phase spectrum* of f
- notation: $F = \mathcal{F}(f)$ means F is the Fourier transform of f; as for Laplace transforms we usually use uppercase letters for the transforms ($e.g.$, $x(t)$ and $X(\omega)$, $h(t)$ and $H(\omega)$, etc.)

Fourier transform and Laplace transform

Laplace transform of f

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Fourier transform of f

$$G(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

very similar definitions, with two differences:

- Laplace transform integral is over $0 \leq t < \infty$; Fourier transform integral is over $-\infty < t < \infty$
- Laplace transform: s can be any complex number in the region of convergence (ROC); Fourier transform: $j\omega$ lies on the imaginary axis

therefore,

- if $f(t) = 0$ for $t < 0$,
  - if the imaginary axis lies in the ROC of $\mathcal{L}(f)$, then

$$G(\omega) = F(j\omega),$$

    $i.e.$, the Fourier transform is the Laplace transform evaluated on the imaginary axis
  - if the imaginary axis is not in the ROC of $\mathcal{L}(f)$, then the Fourier transform doesn't exist, but the Laplace transform does (at least, for all s in the ROC)

- if $f(t) \neq 0$ for $t < 0$, then the Fourier and Laplace transforms can be very different

examples

- one-sided decaying exponential

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases}$$

Laplace transform: $F(s) = 1/(s+1)$ with ROC $\{s \mid \Re s > -1\}$

Fourier transform is

$$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt = \frac{1}{j\omega + 1} = F(j\omega)$$

- one-sided growing exponential

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{t} & t \geq 0 \end{cases}$$

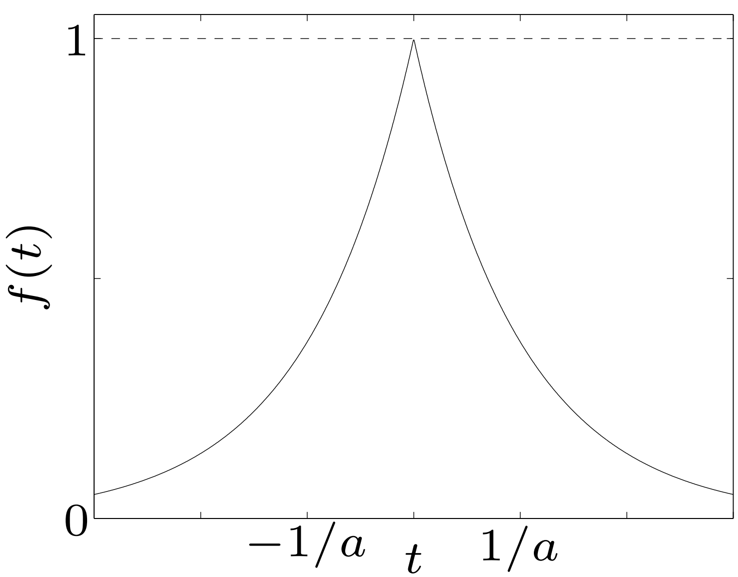
Laplace transform: $F(s) = 1/(s-1)$ with ROC $\{s \mid \Re s > 1\}$

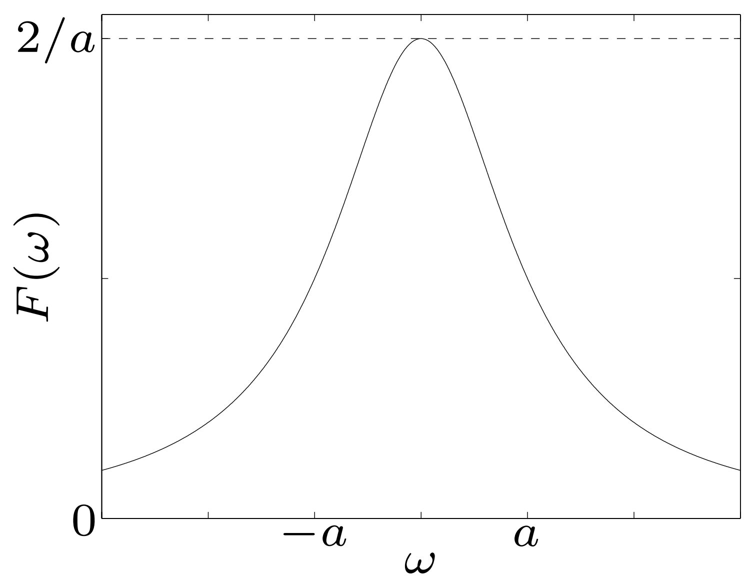
f doesn't have a Fourier transform

Examples

double-sided exponential: $f(t) = e^{-a|t|}$ (with $a > 0$)

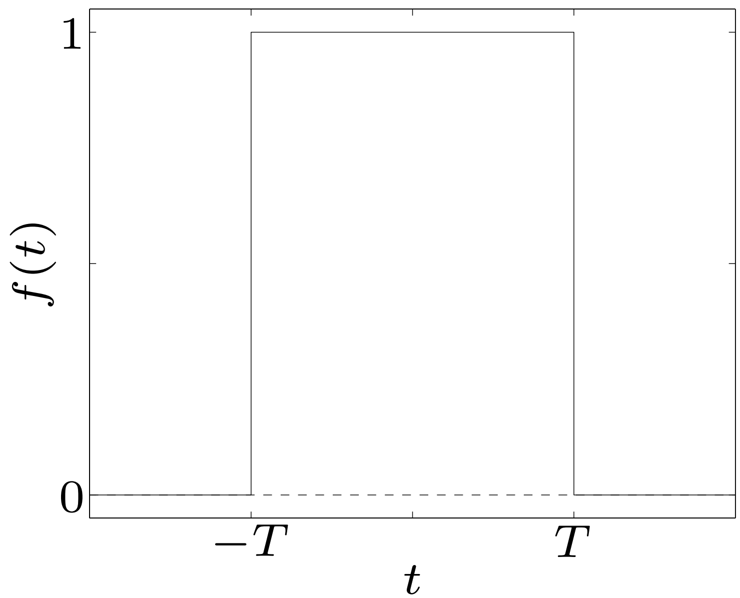
$$
\begin{aligned}
F(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt &= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\
&= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\
&= \frac{2a}{a^2 + \omega^2}
\end{aligned}
$$

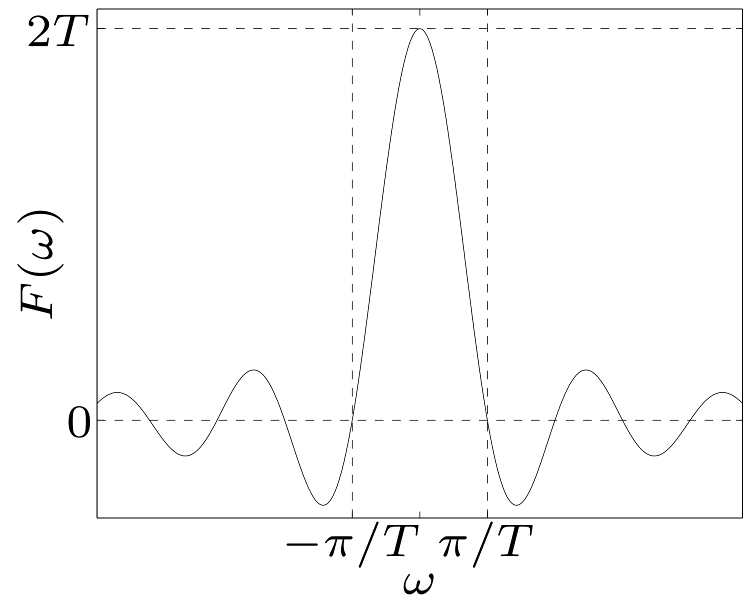




rectangular pulse: $f(t) = \begin{cases} 1 & -T \leq t \leq T \\ 0 & |t| > T \end{cases}$

$$F(\omega) = \int_{-T}^{T} e^{-j\omega t}\, dt = \frac{-1}{j\omega}\left(e^{-j\omega T} - e^{j\omega T}\right) = \frac{2\sin \omega T}{\omega}$$



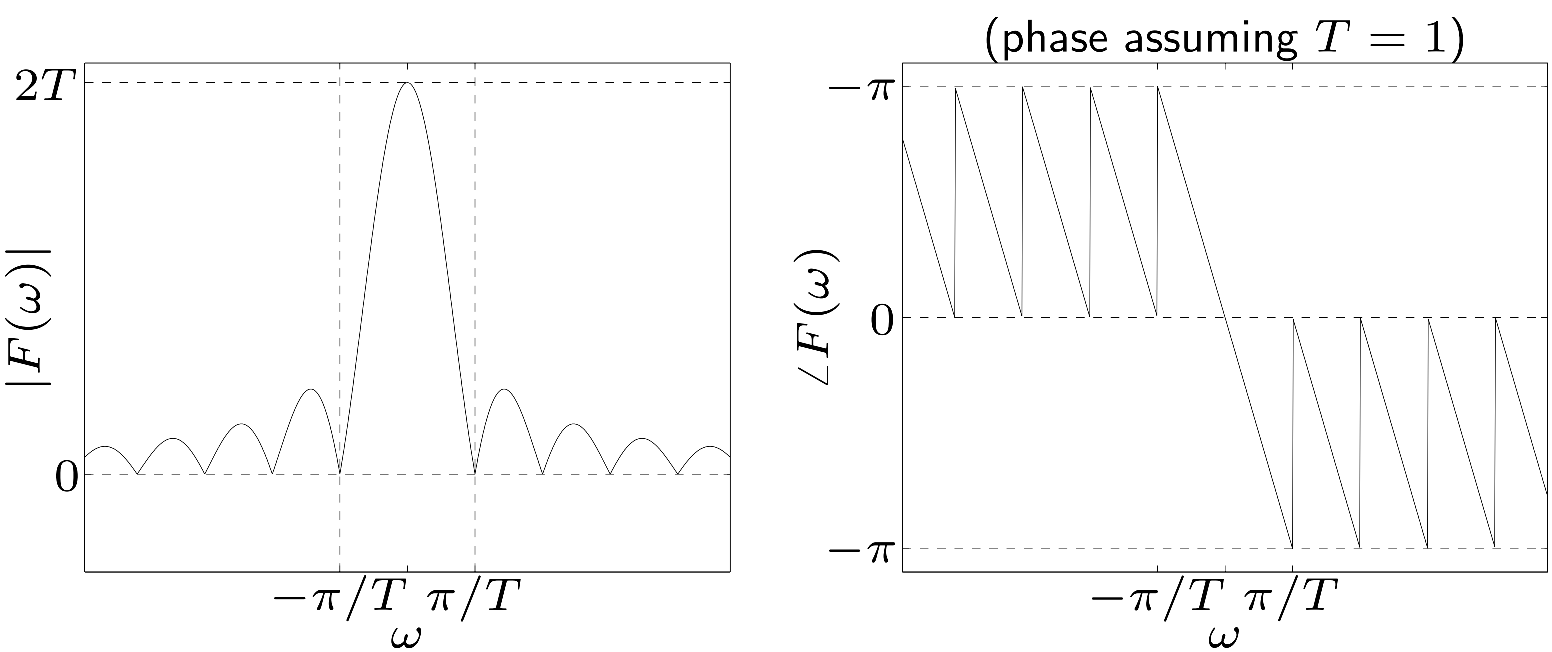


unit impulse: $f(t) = \delta(t)$

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = 1$$

shifted rectangular pulse: $f(t) = \begin{cases} 1 & 1-T \leq t \leq 1+T \\ 0 & t < 1-T \text{ or } t > 1+T \end{cases}$

$$\begin{aligned} F(\omega) = \int_{1-T}^{1+T} e^{-j\omega t}\, dt &= \frac{-1}{j\omega}\left(e^{-j\omega(1+T)} - e^{-j\omega(1-T)}\right) \\ &= \frac{-e^{-j\omega}}{j\omega}\left(e^{-j\omega T} - e^{j\omega T}\right) \\ &= \frac{2\sin\omega T}{\omega}e^{-j\omega} \end{aligned}$$



Step functions and constant signals

by allowing impulses in $\mathcal{F}(f)$ we can define the Fourier transform of a step function or a constant signal

unit step

what is the Fourier transform of

$$f(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \; ?$$

the Laplace transform is $1/s$, but the imaginary axis is not in the ROC, and therefore the Fourier transform is *not* $1/j\omega$

in fact, the integral

$$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-j\omega t}\,dt = \int_{0}^{\infty} \cos\omega t\,dt - j\int_{0}^{\infty} \sin\omega t\,dt$$

is not defined

however, we can interpret f as the limit for $\alpha \to 0$ of a one-sided decaying exponential

$$g_\alpha(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0, \end{cases}$$

$(\alpha > 0)$, which has Fourier transform

$$G_\alpha(\omega) = \frac{1}{a + j\omega} = \frac{a - j\omega}{a^2 + \omega^2} = \frac{a}{a^2 + \omega^2} - \frac{j\omega}{a^2 + \omega^2}$$

as $\alpha \to 0$,

$$\frac{a}{a^2 + \omega^2} \to \pi\delta(\omega), \qquad -\frac{j\omega}{a^2 + \omega^2} \to \frac{1}{j\omega}$$

let's therefore *define* the Fourier transform of the unit step as

$$F(\omega) = \int_0^\infty e^{-j\omega t}\, dt = \pi\delta(\omega) + \frac{1}{j\omega}$$

negative time unit step

$$f(t) = \begin{cases} 1 & t \le 0 \\ 0 & t > 0 \end{cases}$$

$$F(\omega) = \int_{-\infty}^{0} e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{j\omega t}\, dt = \pi\delta(\omega) - \frac{1}{j\omega}$$

constant signals: $f(t) = 1$

f is the sum of a unit step and a negative time unit step:

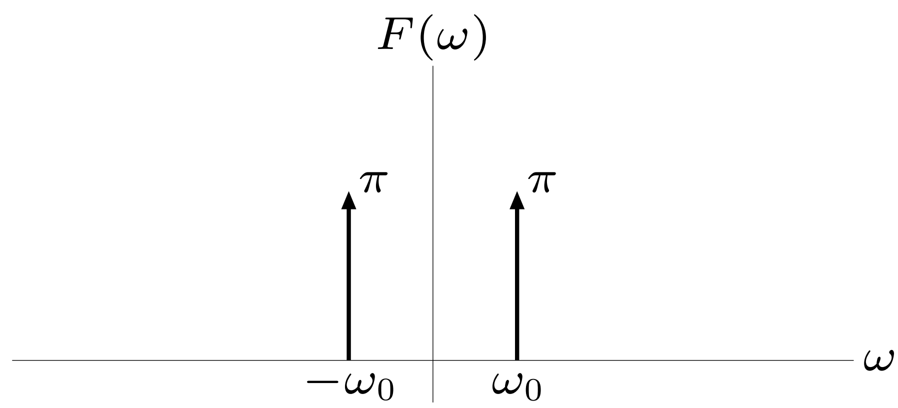
$$F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-j\omega t}\, dt = 2\pi\delta(\omega)$$

Fourier transform of periodic signals

similarly, by allowing impulses in $\mathcal{F}(f)$, we can define the Fourier transform of a periodic signal

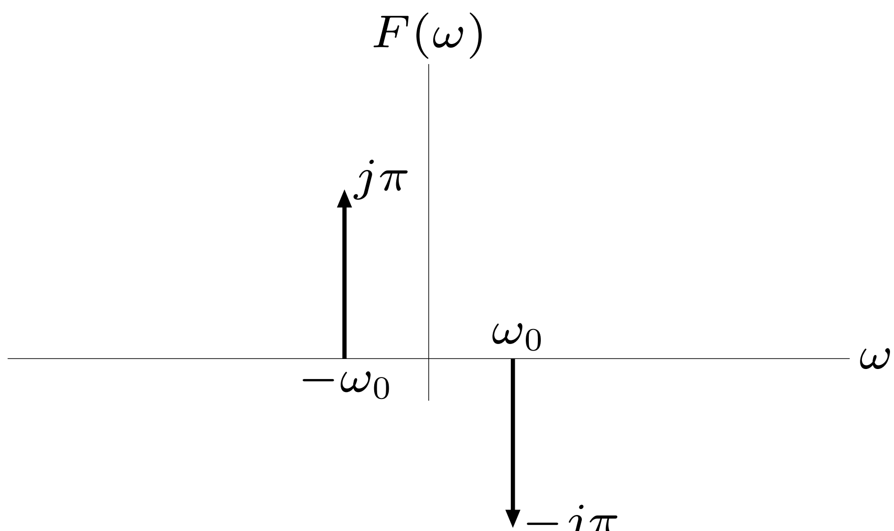
sinusoidal signals: Fourier transform of $f(t) = \cos \omega_0 t$

$$\begin{aligned} F(\omega) &= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-j\omega t}\, dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t} dt \\ &= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \end{aligned}$$



Fourier transform of $f(t) = \sin \omega_0 t$

$$
\begin{aligned}
F(\omega) &= \frac{1}{2j} \int_{-\infty}^{\infty} \left(e^{j\omega_0 t} - e^{-j\omega_0 t} \right) e^{-j\omega t}\, dt \\
&= \frac{1}{2j} \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt + -\frac{1}{2j} \int_{-\infty}^{\infty} e^{-j(\omega_0+\omega)t} dt \\
&= -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)
\end{aligned}
$$

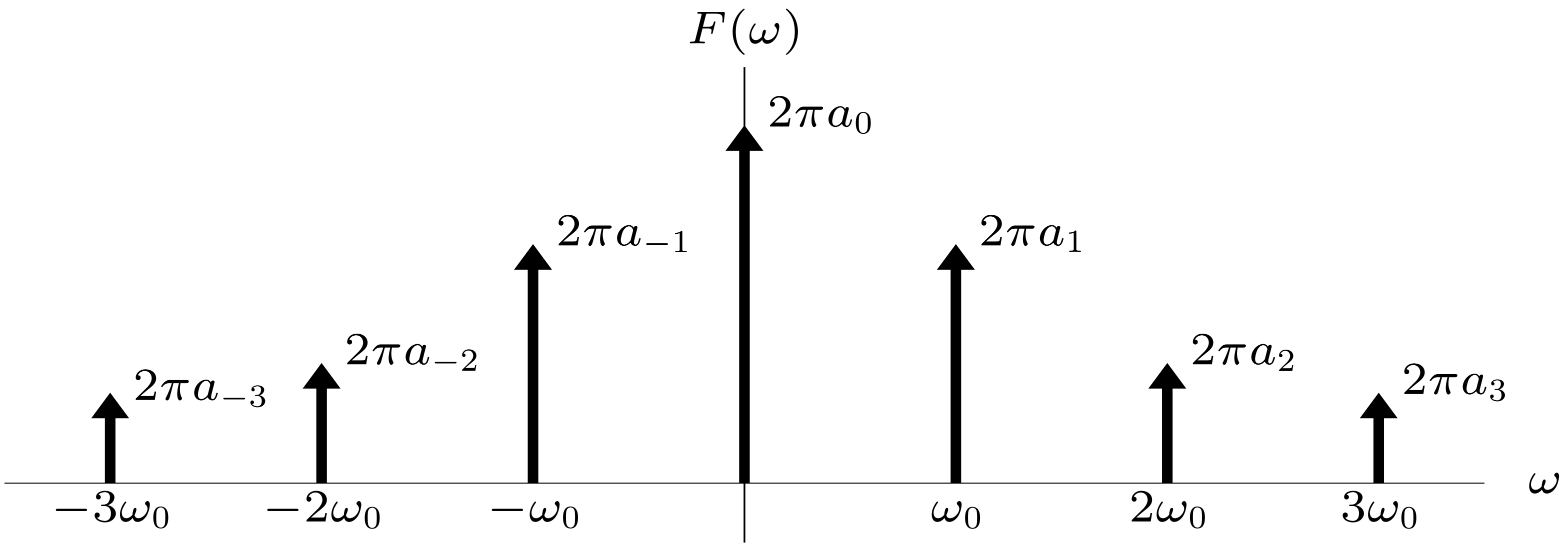


periodic signal $f(t)$ with fundamental frequency ω_0

express f as Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$F(\omega) = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} e^{j(k\omega_0 - \omega)t}\, dt = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$



Properties of the Fourier transform

linearity	$af(t) + bg(t)$	$aF(\omega) + bG(\omega)$
time scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$
time shift	$f(t-T)$	$e^{-j\omega T}F(\omega)$
differentiation	$\frac{df(t)}{dt}$	$j\omega F(\omega)$
	$\frac{d^k f(t)}{dt^k}$	$(j\omega)^k F(\omega)$
integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
multiplication with t	$t^k f(t)$	$j^k \frac{d^k F(\omega)}{d\omega^k}$
convolution	$\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau$	$F(\omega)G(\omega)$
multiplication	$f(t)g(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\widetilde{\omega})G(\omega-\widetilde{\omega})\, d\widetilde{\omega}$

Examples

sign function: $f(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t < 0 \end{cases}$

write f as $f(t) = -1 + 2g(t)$, where g is a unit step at $t = 0$, and apply linearity

$$F(\omega) = -2\pi\delta(\omega) + 2\pi\delta(\omega) + \frac{2}{j\omega} = \frac{2}{j\omega}$$

sinusoidal signal: $f(t) = \cos(\omega_0 t + \phi)$

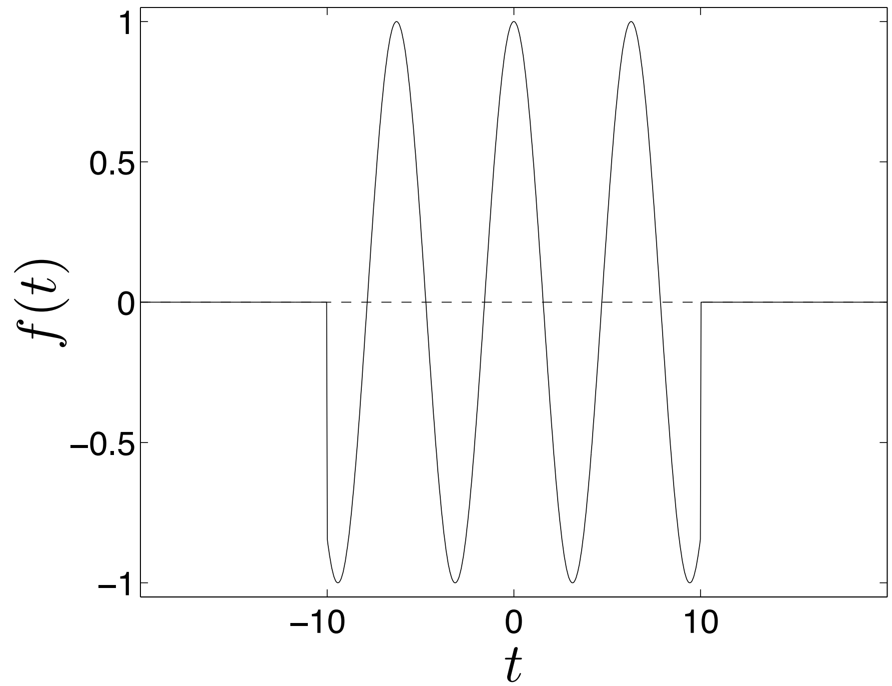
write f as

$$f(t) = \cos(\omega_0(t + \phi/\omega_0))$$

and apply time shift property:

$$F(\omega) = \pi e^{j\omega\phi/\omega_0}\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

pulsed cosine: $f(t) = \begin{cases} 0 & |t| > 10 \\ \cos t & -10 \le t \le 10 \end{cases}$



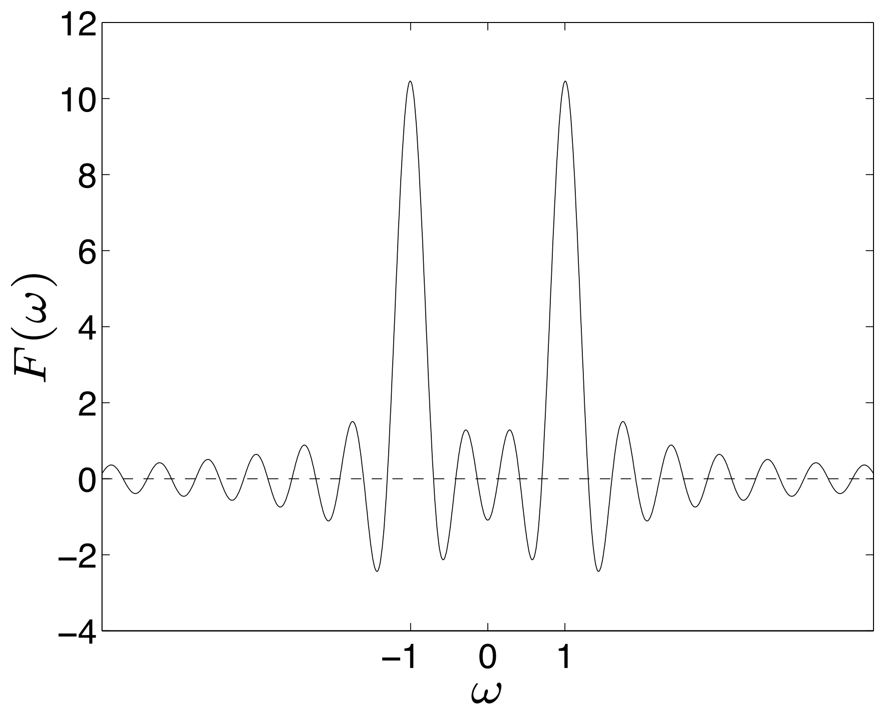
write f as a product $f(t) = g(t)\cos t$ where g is a rectangular pulse of width 20 (see page 12-7)

$$\mathcal{F}(\cos t) = \pi\delta(\omega - 1) + \pi\delta(\omega + 1), \quad \mathcal{F}(g(t)) = \frac{2\sin 10\omega}{\omega}$$

now apply multiplication property

$$F(j\omega) = \int_{-\infty}^{\infty} \frac{\sin 10\widetilde{\omega}}{\widetilde{\omega}} \left(\delta(\omega - \widetilde{\omega} - 1) + \delta(\omega - \widetilde{\omega} + 1)\right) d\widetilde{\omega}$$

$$= \frac{\sin(10(\omega - 1))}{\omega - 1} + \frac{\sin(10(\omega + 1))}{\omega + 1}$$



The inverse Fourier transform

if $F(\omega)$ is the Fourier transform of $f(t)$, *i.e.*,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$$

then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega$$

let's check

$$\begin{aligned}
\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\left(\int_{\tau=-\infty}^{\infty} f(\tau)e^{-j\omega\tau}\right)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{\tau=-\infty}^{\infty} f(\tau)\left(\int_{\omega=-\infty}^{\infty} e^{-j\omega(\tau-t)}d\omega\right)d\tau \\
&= \int_{-\infty}^{\infty} f(\tau)\delta(\tau-t)d\tau \\
&= f(t)
\end{aligned}$$